ON THE FOURIER COEFFICIENTS OF POINCARÉ SERIES OF DIMENSION \(-2\)

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Abstract

Full Text

MATHEMATICS

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ON THE FOURIER COEFFICIENTS OF POINCARÉ SERIES OF DIMENSION −2

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Let $\Gamma(1), \Gamma(N)$ be the well-known groups of integral unimodular matrices of the second order ([1]), $A = \begin{pmatrix} a_0 & a_3 \\ a_1 & a_2 \end{pmatrix} \in \Gamma(1)$, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathfrak{S}(A)$ the system of matrices $M = \begin{pmatrix} m_0 & m_3 \\ m_1 & m_2 \end{pmatrix}$ from the adjoining class $A\Gamma(N)$, for which the pairs (m_1, m_2) are all distinct; $\tau = x + iy$ is a complex variable, $y > 0$; l, ν are positive integers; $T \ll S$, where $S > 0$, is I. M. Vinogradov' s notation indicating that $|T| \leq cS$; in our work the c' s are absolute constants; $\tau(l)$ is the number of divisors of l;

$$W_{m_1}(l, \nu) = \sum_{\substack{j \pmod{m_1 N} \\ j \equiv a_2 (\bmod N),\ (j, m_1) = 1}} \exp\left(2\pi i \frac{jl + h\nu}{m_1 N}\right)$$

is a Kloosterman sum, where $hj \equiv 1 + m_1 a_3 \pmod{m_1 N}$, $h \equiv a_0 \pmod{N}$. This sum can easily be expressed in the form of the ordinary Kloosterman sum $S(u, v, \lambda, \Lambda, q)$ ([2]), but precisely this form is convenient for us. $J_1(x)$ is the Bessel function of index 1. Let

$$S(A, N, l, \nu) = \sum_{\substack{m_1 \equiv a_1 (\bmod N) \\ m_1 \neq 0}} \frac{W_{m_1}(l, \nu)}{|m_1|} J_1\left(4\pi \frac{\sqrt{l}\sqrt{\nu}}{N|m_1|}\right). \tag{1}$$

In the present note we estimate the sum $S(A, N, l, \nu)$ with respect to N, ν, l. The estimate obtained is of interest because it cannot be improved with respect to l. This fact is connected with deep results of A. Weil and Eichler.

Theorem 1. Let $\left(\frac{l}{(l, N)}, N\right) = 1$. Then

$$S(A, N, l, \nu) \ll \tau(l) \nu^{5/6} N^{7/4}. \tag{2}$$

If $A = I$, then one can obtain an unimprovable estimate also with respect to ν.

Theorem 2. Let $\left(\frac{l}{(l,N)}, N\right) = \left(\frac{\nu}{(\nu,N)}, N\right) = 1$. Then

$$S(I, N, l, \nu) \ll \tau(l)\tau(\nu)N^6.$$

The proofs of the theorems are based on a series of lemmas and some facts from the theory of modular forms.

Let

$$G_{-2}(\tau, A, N, \nu) = \sum_{M \subset \mathfrak{S}(A)} \frac{\exp\left(2\pi i \frac{M\tau}{N}\nu\right)}{(m_1\tau + m_2)^2}$$

be a Poincaré series of dimension −2 ([3]). It is known ([2]) that $G_{-2}(\tau, A, N, \nu)$ is an entire parabolic form of dimension −2, belonging to the group $\Gamma(N)$, and the following expansion into a Fourier series in a neighborhood is valid-

at the cusp at infinity $\binom{2}{3}$:

$$G_{-2}(\tau, A, N, v) = \sum_{l=1}^{\infty} a_l \exp\left(2\pi i \frac{\tau}{N} l\right),$$

$$a_l = \delta_l^v + \frac{-2\pi}{N\sqrt{v}} l^{1/2} S(A, N, l, v), \tag{3}$$

where δ_l^v is the Kronecker symbol.

As is known ([4]), the space of integral parabolic forms $\mathfrak{S}(N)$ of type $\{\Gamma(N), -2\}$ (we denote its dimension by g) is the direct sum of the subspaces $\mathfrak{S}(t, \chi, N)$, consisting of integral parabolic forms of divisor t and character χ. On the space $\mathfrak{S}(N)$ one introduces the Petersson inner product (f, g) ([5]). Each of the subspaces $\mathfrak{S}(t, \chi, N)$ has an orthogonal basis consisting of eigenfunctions of all Hecke operators T_n, where $(n, N) = 1$, acting on this subspace, and the eigenfunctions from different bases are orthogonal ([5]). From the bases of the subspaces $\mathfrak{S}(t, \chi, N)$ we form an orthogonal basis of the space $\mathfrak{S}(N)$. Let this basis consist of the functions

$$f_i(\tau) = \sum_{\substack{n=1 \\ (n,N)=t}}^{\infty} \tau_i(n) \exp\left(2\pi i \frac{\tau}{N} n\right) \quad (i = 1, 2, \dots, g). \tag{4}$$

We may assume that $\tau_i(t) = 1$. Then from the results of ([6−9]) it follows that

$$\tau_i(l) < \tau(l)\sqrt{l}, \tag{5}$$

where $\left(\frac{l}{(l,N)}, N\right) = 1$.

Lemma 1.

$$(f_i(\tau), f_i(\tau)) > \frac{1}{4\pi e^{4\pi}}. \tag{6}$$

The proof is analogous to the proof of Lemma 2 ([10]), taking into account that as a fundamental domain D of the group $\Gamma(N)$ one may choose a domain in the τ-plane containing the domain of the upper half-plane bounded by the straight lines $x = 1,\ x = (2N-1)/2,\ y = 1$.

Lemma 2.

$$a_l \ll \frac{l v^{1/3}}{N^{13/4}}. \tag{7}$$

The proof is trivial and uses the simplest properties of the function $J_1(x)$ in the following estimate, which follows from the results of Salié ([11]):

$$W_{m_1}(l, v) \ll |m_1|^{3/4} v^{1/3}.$$

Lemma 3. Let $B \in \Gamma(1)$. Then

$$G_{-2}(\tau, A, N, v)/B = G_{-2}(\tau, AB, N, v).$$

For the proof see ([2]).

Lemma 4.

$$(G_{-2}(\tau, A, N, v), G_{-2}(\tau, A, N, v)) \ll \frac{v^{1/3}}{N^{2/3}}. \tag{8}$$

Proof. Let D_1 be a region of width N, obtained by shifting the fundamental domain D_0 of the group $\Gamma(1)$ (one may assume that D_0 consists of points τ satisfying $|\tau| > 1,\ x < {}^1/_2$ (!)) to the right by $1, 2, \dots, N$ and taking the union of the resulting regions; let P be the part of the upper half-plane of the τ-plane bounded by the straight lines $x = -{}^1/_2,\ x = (2N-1)/2,\ y = \sqrt{3}/2$. Clearly, $D_1 \subset P$. As was shown in ([10]), one can choose a system of matrices $\sigma_i^{(0)}$ from $\Gamma(1)$, in number

$$r = N^2 \prod_{p|N} \left(1 - \frac{1}{p^2}\right),$$

such that

$$(G_{-2}(\tau, A, N, \nu), G_{-2}(\tau, A, N, \nu)) =$$

$$= \sum_{i=1}^{r} \iint_{D_1} G_{-2}(\tau, A, N, \nu)/\sigma_i^{(0)} \cdot \overline{G_{-2}(\tau, A, N, \nu)/\sigma_i^{(0)}}\, dx\, dy.$$

Replacing in each integral the region D_1 by P, and then applying Lemma 3, the expansion (3), and termwise integration, we obtain

$$\iint_{D_1} G_{-2}(\tau, A, N, \nu)/\sigma_i^{(0)} \cdot \overline{G_{-2}(\tau, A, N, \nu)/\sigma_i^{(0)}}\, dx\, dy \ll$$

$$\ll \sum_{l=1}^{\infty} |a_l|^2 \int_{\sqrt{3}/2}^{\infty} \exp\left(-4\pi \frac{y}{N} l\right) dy.$$

Using (7) and carrying out elementary estimates, we obtain (8).

Proof of Theorem 1. Expand $G_{-2}(\tau, A, N, \nu)$ in the orthonormal basis (4):

$$G_{-2}(\tau, A, N, \nu) = \sum_{i=1}^{g} \alpha_i f_i(\tau); \tag{9}$$

whence

$$a_l = \sum_{i=1}^{g} \alpha_i \tau_i(l).$$

Clearly,

$$(G_{-2}(\tau, A, N, \nu), G_{-2}(\tau, A, N, \nu)) = \sum_{i=1}^{g} |\alpha_i|^2 (f_i(\tau), f_i(\tau)).$$

It is known ([1]) that $g \ll N^3$. Using (5) and (6), we easily derive

$$a_l \ll \tau(l) \sqrt{l N^3 \left(G_{-2}(\tau, A, N, \nu), G_{-2}(\tau, A, N, \nu)\right)}\,.$$

The assertion of the theorem now follows from (8).

The proof of Theorem 2 is carried out by means of formula (110) ([2]) and (5), (6).

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Note: Figure translations are in progress. See original paper for figures.