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Abstract

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MATHEMATICS

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ON THE CONCEPT OF A RANDOM NORMED SPACE

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§ 1. In the proposed article we consider a class of linear spaces in which, instead of the concept of the norm of a vector, the concept of the distribution of the norm is introduced. We shall call these spaces random normed spaces (r.n.s.). A similar generalization of metric space was first carried out by Menger ([1]) and led to the theory of random metrics. The principal point in the axiomatics of a random metric is the "triangle inequality." Menger proposed associating with a space with a random metric a certain function T of two real variables. By means of this function T the triangle inequality is introduced. Subsequently Wald ([2]), noting certain inconveniences of Menger' s axiomatics, proposed his own triangle inequality. Then Schweizer and Sklar ([3]) observed that Wald' s space is a Menger space with a special function T. However, a Menger space with this T, in general, is not a Wald space. Therefore Wald' s theory does not follow from Menger' s theory.

In the present work an axiomatics is constructed that includes the theories of Menger and Wald as special cases. It is necessary only to remember that in our case the exposition is conducted for a narrower class of spaces. A reformulation of the proposed axiomatics for random metrics, however, presents no difficulties. In accordance with the aim of this work, an additional axiom ($\mu 5$) is introduced, expressing in probabilistic interpretation the following: the event {the norm of the sum of two vectors is not less than $x + y$} entails that either {the norm of the first element is not less than x}, or {the norm of the second element is not less than y}. The concept of an r.n.s. is quite general. We have shown that an arbitrary countably normed space can be regarded as an r.n.s. In the present work the question of the completion of an r.n.s. is also studied.

§ 2. Let $\mathfrak{B}$ be the set of all nonincreasing left-continuous functions $\xi(x)$, defined on the whole real axis R, such that $\xi(x) = 1$ if $x \le 0$, and $\xi(\infty) = 0$. Let B be some subset of the set $\mathfrak{B}$. For a given function α, defined on $\mathfrak{B} \times \mathfrak{B}$ with values in $\mathfrak{B}$, form the set

$$B[\alpha] = \bigcup_{n=0}^{\infty} B_n[\alpha],$$

where $B_0[\alpha] = B$,

$$B_n[\alpha] = B_{n-1}[\alpha] \cup \alpha(B_{n-1}[\alpha] \times B_{n-1}[\alpha]).$$

In the set $\mathfrak{B}$ introduce an order relation, taking $\xi \leq \eta$ if $\xi(x) \leq \eta(x)$ for every $x \in R$. We shall call the function α a B-function if for all ξ, η, ζ from $B[\alpha]$:

$$(\mu 1) \quad \alpha(\xi, \eta) = \alpha(\eta, \xi).$$

$$(\mu 2) \quad \alpha(\Delta, \xi) = \xi,$$

where Δ is a function from $\mathfrak{B}$ such that $\Delta(x) = 0$ for $x > 0$.

$$(\mu 3) \quad \alpha(\xi, \eta) \geq \alpha(\xi_1, \eta_1),$$

if $\xi \geq \xi_1,\ \eta \geq \eta_1$.

$$(\mu 4) \quad \alpha(\alpha(\xi, \eta), \zeta) = \alpha(\xi, \alpha(\eta, \zeta)).$$

$$(\mu 5) \quad \alpha(\xi, \eta \mid x) \leq \inf_{t \in [0,1]} \min\{\xi(tx) + \eta((1-t)x),\, 1\}, \quad x \in R.$$

Here $\alpha(\xi, \eta/x)$ is the value of the function $\alpha(\xi, \eta)$ at the point $x \in R$.

Definition. An R. n. s. is a triple (Ω, f, μ), where Ω is some linear space over the field Λ of complex or real numbers, f is a mapping of Ω into $\mathfrak{B}$: $\varphi \in \Omega \to f(\varphi) = \|\varphi\| = \|\varphi; \cdot\| \in \mathfrak{B}$, and μ is some $f(\Omega)$-function. In addition, the following axioms are satisfied:

I. $\|\varphi\| = \Delta$ if and only if $\varphi = \theta$ (θ is the zero element of Ω).

II. $\|a\varphi; x\| = \|\varphi; x/|a|\|$ for every $\varphi \in \Omega$ and every $a \in \Lambda$;

III. $\|\varphi + \psi\| \leq \mu(\|\varphi\|, \|\psi\|)$, $\varphi \in \Omega,\ \psi \in \Omega$.

An R. n. s. is a generalization of ordinary normed spaces. The function $\|\varphi; x\|$ may be interpreted as the probability that the norm of the vector φ is not less than x. An ordinary normed space with norm $p(\varphi)$ becomes an R. n. s. if as μ one takes an arbitrary $f(\Omega)$-function such that

$$\mu(\xi, \eta \mid x) \geq \mu_0(\xi, \eta \mid x) = \inf_{t \in [0,1]} \max\{\xi(tx), \eta((1-t)x)\},$$

and sets

$$\|\varphi; x\| = \begin{cases} 1, & \text{if } x \le p(\varphi), \\ 0, & \text{if } x > p(\varphi). \end{cases}$$

If as μ one takes

$$\mu(\xi, \eta \mid x) = \eta(x) - \int_0^x \xi(x - s)\, d\eta(s),$$

we obtain Wald' s space. It can also be shown that Menger' s theory is included in our theory.

If μ' and μ'' are such that $\mu'(\xi, \eta) \le \mu''(\xi, \eta)$ for arbitrary $\xi \in \mathfrak{B}$, $\eta \in \mathfrak{B}$, then the space (Ω, f, μ') is also a space (Ω, f, μ''). This remark leads to the following problem. Let Ω be a linear space and f some mapping of Ω into $\mathfrak{B}$ such that, for the elements of the image $f(\Omega)$, axioms I and II are satisfied. It is required, among all $f(\Omega)$-functions μ for which (Ω, f, μ) is an R. n. s., to find the "best" one. Introduce on the set of all $f(\Omega)$-functions an order relation, taking $\mu' \le \mu''$ if $\mu'(\xi, \eta) \le \mu''(\xi, \eta)$ for arbitrary $\xi \in \mathfrak{B}$, $\eta \in \mathfrak{B}$. Then this problem is naturally understood as the problem of finding a minimal element in the set of all $f(\Omega)$-functions μ for which (Ω, f, μ) is an R. n. s. One can prove the following result, which is an analogue of Theorem 4 ([4]).

Theorem 1. Let Ω be a linear space and f some mapping of Ω into $\mathfrak{B}$ such that, for the elements of the image $f(\Omega)$, axioms I and II are satisfied. Let J be the set of all $f(\Omega)$-functions μ such that: 1) (Ω, f, μ) is an R. n. s.; 2) properties $(\mu 1) - (\mu 5)$ are satisfied on all of $\mathfrak{B}$; 3) $\mu(\cdot, \eta \mid x)$ is continuous with respect to the convergence of monotone decreasing networks on $\mathfrak{B}$ in the first argument when the second is fixed (if convergence is understood as pointwise convergence). Then J has a minimal element.

§ 3. Introduce for consideration the sets

$$U_{\varepsilon,\delta} = \{\varphi \in \Omega : \|\varphi; \delta\| < \varepsilon\},$$

$0 < \varepsilon < 1,\ 0 < \delta < \infty$. The system $\mathfrak{A} = \{U_{\varepsilon,\delta}\}$ turns out to be a defining system of neighborhoods of zero in Ω, and, consequently, Ω may be regarded as a topological linear space. In Ω the first axiom of countability is satisfied; therefore the topology can be described by means of countable convergent sequences. In this case a sequence φ_n converges to an element $\varphi \in \Omega$ if and only if $\|\varphi - \varphi_n; x\| \to 0$ for every $x > 0$.

Theorem 2. If (Ω, f, μ) is an R. n. s., then $\mu(\|\varphi\|, \|\psi\|)$ is a continuous function of φ and ψ in the sense that, if $\varphi_n \to \varphi,\ \psi_m \to \psi$, then

$$\mu(\|\varphi_n\|, \|\psi_m\| \mid x) \to \mu(\|\varphi\|, \|\psi\| \mid x)$$

$(n, m \to \infty)$ for every point of continuity of the function $\mu(\|\varphi\|, \|\psi\|)$.

It follows from this, in particular, that $f(\varphi) = \|\varphi\|$ is a continuous function of φ in the same sense. The general definition of a bounded set in a topological linear space gives the following criterion of boundedness—

···of boundedness of a set in an r.n.s.: a set $A \subset \Omega$ is bounded if and only if there exists a function $\xi \in \mathfrak{B}$ such that $\|\varphi\| \leq \xi$ for every $\varphi \in A$.

§ 4. We shall next set forth some results on the completion of an r.n.s. As usual, a sequence φ_n of elements of an r.n.s. Ω will be called **fundamental** if $\varphi_n - \varphi_m \to \theta$ $(n, m \to \infty)$. Ω is a complete space if every fundamental sequence φ_n of elements of Ω converges. As is known, every normed space can be isometrically embedded in some complete normed space. It is of interest to investigate the question of such an embedding in the case of an r.n.s. By an isometry here one should naturally understand a mapping that preserves the functions $\|\varphi\|$. In what follows, by a completion we mean such an isometry into some complete space.

Theorem 3. *Every r.n.s.* (Ω, f, μ) *has some completion* (Ω', f', μ'), *if* μ *is a* $\mathfrak{B}$*-function continuous with respect to the topology* $\mathfrak{B}$ *as a topology induced by pointwise convergence.*

If the r.n.s. Ω satisfies an axiom stronger than axiom I, namely, if for every $\varphi \in \Omega$

$$\|\varphi; +0\| = \begin{cases} 1, & \text{if } \varphi \neq \theta, \\ 0, & \text{if } \varphi = \theta, \end{cases}$$

then the completion Ω', generally speaking, no longer satisfies this axiom. One can verify this by constructing an appropriate example.

Under the conditions of Theorem 3 the completion is carried out in such a way that the completed space inherits the $f(\Omega)$-function of the original space. The problem of completion may also be posed with simultaneous change of μ.

Theorem 4. *For every r.n.s.* (Ω, f, μ) *there exists some completion* (Ω', f', μ'), *where* $\mu' \geq \mu$.

Let us give an example of an infinite-dimensional r.n.s. Let $\{\xi_i\}_1^\infty$ be a sequence of functions from $\mathfrak{B}$, distinct from Δ. Let m be the set of sequences $\varphi = (\varphi_1, \varphi_2, ...)$ of complex numbers such that

$$\sup_i \xi_i\,(x/|\varphi_i|) \to 0 \quad \text{as } x \to \infty.$$

It is not difficult to show that m is a linear space with respect to the natural vector operations on the elements φ. Define f by putting

$$\|\varphi; x\| = \sup_i \xi_i \left(\frac{x}{|\varphi_i|}\right).$$

Then (m, f, μ_0) is a complete locally convex space.

§ 5. In the present paragraph it will be shown that every countably normed space can be regarded as an r.n.s. Let (Φ, q) be an arbitrary countably normed space. Here q is a multinorm on Φ: $q = \{|\cdot|_1, |\cdot|_2, ...\}$. We may assume that the prenorms $|\cdot|_k$, $k = 1, 2, ...$, are such that $|\cdot|_1 \leq |\cdot|_2 \leq \cdots$. Adjoin formally to this sequence of prenorms the zero prenorm $|\cdot|_0$, setting $|\varphi|_0 = 0$ for every $\varphi \in \Phi$. Let τ be an arbitrary random variable taking nonnegative integer values, with $P(\tau \geq N) > 0$, whatever N may be. Since τ is given, the function $\xi(x) = P(\tau \geq x)$ is known to us. Associate with each element $\varphi \in \Phi$ the function defined for every $x \in R$:

$$\|\varphi; x\| = P\left(|\varphi|_\tau \geq x\right).$$

If we introduce the function

$$\varphi[x] = \begin{cases} \min\{n : \ |\varphi|_n \geq x\}, & \text{if } \{n : \ |\varphi|_n \geq x\} \neq \emptyset; \\ \infty, & \text{otherwise,} \end{cases}$$

then it is not difficult to show that

$$\|\varphi; x\| = \xi(\varphi[x]). \tag{*}$$

Theorem 5. Let (Φ, q) be an arbitrary countably normed space, $q = \{|\cdot|_0, |\cdot|_1, ...\}$ a multinorm defined on Φ. Then the triple (Φ, f_τ, μ_0) is an r.n.s. (here f_τ is the mapping defined by means of $(*)$). Moreover, the mapping that sends each element φ of the original space (Φ, q) to the same element, regarded as an element of the r.n.s. (Φ, f_τ, μ_0), realizes an algebraic and topological isomorphism between these spaces.

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