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Abstract

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ASYMPTOTIC REPRESENTATIONS IN SOME PROBLEMS FOR TWO-DIMENSIONAL RANDOM WALKS

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In (^{1}, ^{2}), in order to obtain limit theorems in boundary-value problems for sums of independent summands, the asymptotic behavior of the solution of the equation

$$P^y_{x,t} - \int_{-\infty}^{x} P^y_{x-u,t-1}\, dF(u) = f(x,y,t), \tag{1}$$

was studied as x, t (and sometimes also y) increase without bound. In (1), $F(u)$ is a distribution function; the solution is sought in the class of bounded, nonnegative functions; $f(x,y,t)$ is determined by the probabilistic meaning of the function $P^y_{x,t}$. In the present note the results of (^{1}, ^{2}) are generalized to the equation

$$P^{y,s}_{x,t} - \int_{-\infty}^{x} dF(u) \int_0^t dG(v)\, P^{y,s}_{x-u,t-v} = f(x,y,t,s), \tag{2}$$

where G is also a distribution function, and the asymptotics of the solution is established as $x, t \to \infty$ (and possibly also y, s). At the same time, in contrast to (^{1}, ^{2}), we shall consider the case in which $F(u)$ is lattice-valued.

Solutions of equation (2), for suitable right-hand sides, will describe, for example, limit theorems in the following boundary-value problems for a two-dimensional random walk. Let $\xi^{(1)}_1, \xi^{(1)}_2, \ldots$ be a sequence of independent random variables with distribution function $F(u)$, and let $\xi^{(2)}_1, \xi^{(2)}_2, \ldots$ be a sequence of nonnegative random variables, independent of one another and of the first sequence, with distribution function $G(u)$. Consider in the plane (t, x) the random walk determined by the sequence of points $(0,0)$, $(s^{(2)}_1, s^{(1)}_1)$, $(s^{(2)}_2, s^{(1)}_2)$, ..., where $s^{(i)}_k = \xi^{(i)}_1 + \cdots + \xi^{(i)}_k$, $i = 1, 2$. Denote $\bar{s}^{(1)}_n = \max_{0 \le k \le n} s^{(1)}_k$; η_T is the time of first passage of the level T along the t-axis: $\eta_T = \max\{k : s^{(2)}_k < T\}$. In insurance problems, problems of statistics, and others, the distributions

$$P_{x,t} = \mathbf{P}(\bar{s}^{(1)}_{\eta_t} \ge x),$$

$$
\begin{aligned}
{}_1P^{y,s}_{x,t} &= \mathbf{P}(\bar{s}^{(1)}_{\eta_t} < x, \quad s^{(1)}_{\eta_t+1} = x+y, \quad s^{(2)}_{\eta_t+1} \geq t+s),\\
{}_2P^{y,s}_{x,t} &= \mathbf{P}(\bar{s}^{(1)}_{\eta_t} \geq x, \quad s^{(1)}_{\eta_t+1} = x-y, \quad s^{(2)}_{\eta_t+1} \geq t+s),\\
{}_3P^{y,s}_{x,t} &= \mathbf{P}(s^{(1)}_{\eta_t+1} = x-y, \quad s^{(2)}_{\eta_t+1} \geq t+s),
\end{aligned} \tag{3}
$$

are of interest, where $x, t > 0$, $s \geq 0$, and y may take values of different signs. In many cases it is then possible to determine also the corresponding distributions that are integral with respect to $s^{(1)}_{\eta_t+1}$. New distributions obtained if, in the probabilities (3), the event $s^{(2)}_{\eta_t+1} \geq t+s$ is replaced by the event $s^{(2)}_{\eta_t} < t-s$ are not difficult to find by using relations of the type

$$
\mathbf{P}(\bar{s}^{(1)}_{\eta_t} < x, \quad s^{(1)}_{\eta_t+1} = x-y, \quad s^{(2)}_{\eta_t} < t-u, \quad s^{(2)}_{\eta_t+1} \geq t+v) = {}_1P^{y,u+v}_{x,t-u}.
$$

Denote by ${}_jQ^{y,s}_{x,t}$, $j = 1, 2, 3$, the probabilities obtained by replacing in (3) the event $s^{(1)}_{\eta_t+1} = x-y$ by the event $s^{(1)}_{\eta_t} = x-y$. It turns out to be possible to find asymptotic representations for the transforms with respect to t of the probabilities (3) and ${}_jQ^{y,s}_{x,t}$ $(j = 1, 2, 3)$ (which are solutions of (2)), allowing one to carry out a rather complete asymptotic analysis of the distributions themselves (cf. (2)).

If $G(+0) = 0$, then our random walk can also be regarded as a jump process $\{X_t, t > 0\}$, in which changes occur at the time instants $s^{(2)}_1, s^{(2)}_2, \ldots$, so that, putting $G(t) = 1 - e^{-\lambda t}$, we arrive at a generalized Poisson process.

Let us introduce the necessary notation. Let $f(\lambda) = M\lambda^{\xi^{(1)}_k} = \sum_{-\infty}^{\infty} f_k \lambda^k$, and let (λ_-, λ_+) be the largest interval on which $f(\lambda) < \infty$. If a is some notation connected with the distribution of $\xi^{(2)}_k$, then, when necessary, we shall write $\dot{a}$ if $\xi^{(2)}_k$ is lattice-valued, and $\tilde{a}$ if $\xi^{(2)}_k$ is nonlattice. Accordingly, put $g(\mu) = M\mu^{\xi^{(2)}_k} = \sum_0^{\infty} g_k \mu^k$, $\tilde{g}(\mu) = Me^{\mu\tilde{\xi}^{(2)}_k}$, $\mu_+ = \sup\{\mu : g(\mu) < \infty\}$.

We shall assume that:

I. $\lambda_+ - \lambda_- > 0$.

II. $\dot{\mu}_+ > 1$; $\tilde{\mu}_+ > 0$, $\limsup_{|\mu|\to\infty} |\tilde{g}(i\mu)| < 1$.

The greatest common divisors (g.c.d.) of possible values of $\xi^{(1)}_k$ and $\xi^{(2)}_k$ may, without loss of generality, be regarded as equal to 1; denote the g.c.d.' s of the differences of possible values of $\xi^{(1)}_k$ and $\xi^{(2)}_k$ by d_1 and d_2, respectively, and let $(d_1, d_2) = d$.

We have $f''(\lambda) \geq 0$ throughout the interval (λ_-, λ_+). Therefore there exist at most two real zeros $\lambda_\pm(z)$ $(\lambda_+(z) \geq \lambda_-(z))$ of the function $1 - zf(\lambda)$, defined respectively for $z \in (z_\pm, z_0)$, where $z_\pm = f^{-1}(\lambda_\pm)$, $z_0 = \sup_{(\lambda_-, \lambda_+)} f^{-1}(\lambda)$. Further, the segment $[\lambda_-, \lambda_+]$ always contains a point λ_0 at which $f(\lambda_0) = \inf_{(\lambda_-, \lambda_+)} f(\lambda)$, so that $z_0 = f^{-1}(\lambda_0)$. For $\lambda_0 \in (\lambda_-, \lambda_+)$, the functions $\lambda_\pm(z)$ can be analytically continued to a neighborhood of the point z_0, which is for them a common branch point, at which they form one circular system. Denote by $K_{\delta\pm}$ the domains $\{z_\pm + \delta_1 \leq |z| \leq z_0 + \delta,\ |\arg z| < \delta\}$, where the numbers δ and $\delta_1 = \delta_1(\delta)$ are chosen so that $K_{\delta\pm}$ contain no other singularities of the functions $\lambda_\pm(z)$ except the point z_0, and let $\hat{\lambda}_\pm(z)$ be the functions coinciding with $\lambda_\pm(z)$ at all points of the intervals $[z_\pm, z_0]$ and equal to $\hat{\lambda}_0$ for $z > z_0$. Introduce the functions $W_{z\pm}(\lambda)$ by means of the following analogue of Lemma 1 in (1).

The function

$$W_z(\lambda) = \frac{(1 - zf(\lambda))\lambda}{(\lambda - \lambda_-(z))(\lambda - \lambda_+(z))}$$

for $z \in K'_{\delta+} \cap K'_{\delta-}$ and sufficiently small δ and γ admits a canonical factorization

$$W_z(\lambda) = W_{z+}(\lambda) \cdot W_{z-}(\lambda)$$

with respect to λ in the domain $\hat{\lambda}_-(|z|) - \gamma \leq |\lambda| \leq \hat{\lambda}_+(|z|) + \gamma$, in which the functions $W_{z\pm}(\lambda)$ are representable in the form of absolutely convergent series $\sum_{k=0}^{\infty} \lambda^{\pm k} w_{z\pm}(k)$.

$W_{z\pm}(\lambda)$ are different from zero respectively for $z \in K'_{\delta+}$, $|\lambda| \leq \hat{\lambda}_+(|z|) + \gamma$; $z \in K'_{\delta-}$, $|\lambda| \geq \hat{\lambda}_-(|z|) - \gamma$. In these domains the functions $W_{z\pm}(\lambda)$ may be chosen regular jointly in the variables z and λ.

We now define the domains E_δ and $K_{\delta+}$. Let $\mu_- = 0$ and $\tilde{\mu}_- = -\infty$. Then, in a neighborhood of any point of the interval (μ_-, μ_+), the equation $z = g(\mu)$ is uniquely solvable with respect to $\mu = M(z)$. To define the function $M(z)$ on the whole half-line $[0, \infty]$, set $M(z) = \mu_-$ for $z \in [0, g(\mu_-)]$.

and $M(z) = \mu_+$ for $z \in [g(\mu_+), \infty]$. Further, let $\mu^\pm = M(z_\pm)$, $\mu_0 = M(z_0)$, $M_\delta(z) = \min(M(z) + \delta, \mu_+)$. By $K_{\delta\pm}$ and $\widetilde{K}_{\delta\pm}$ we shall mean, respectively, the domains

$$\{\mu^\pm + \delta_1(\delta) \leq |\mu| \leq M_\delta(z_0),\ |\arg \mu| < \delta\},$$

$$\{\mu^\pm + \delta_1(\delta) \leq \operatorname{Re} \mu \leq M_\delta(z_0),\ |\operatorname{Im} \mu| < \delta\},$$

and by $E_\delta, \widetilde{E}_\delta$ the domains

$$\{|\mu| \leq M_\delta(z_0),\ |\arg \mu| < \pi/d\}, \quad \{\operatorname{Re} \mu \leq M_\delta(z_0)\},$$

from which the points of the segment $[\mu_0, \mu_0 + \delta]$ have been removed.

Now we can formulate the theorems on asymptotic representations for the transforms of the probabilities (3) and ${}^{y,s}_{jQ_{x,t}}$. Denote

$${}_j\dot P_x^{y,s}(\mu) = \sum_{t=1}^{\infty} {}_j\dot p_{x,t}^{y,s}\mu^t, \qquad {}_j\widetilde P_x^{y,s}(\mu) = \int_0^{\infty} {}_j\widetilde P_{x,t}^{y,s} e^{\mu t}\,dt, \tag{4}$$

$${}_j\dot p_x^{y,s}(\mu) = {}_j\dot P_x^{y,s}(\mu) - {}_j\dot P_x^{y,s+1}(\mu)$$

(correspond to the probabilities (3), local with respect to $s^{(2)}_{\eta_t} + 1$),

$$T(a,b) = \frac{a^{-x}b^{y+1}}{\left[b^2 - \lambda_+(g(\mu))\lambda_-(g(\mu))\right] W_{g(\mu)+}(a)W_{g(\mu)-}(b)}\,\frac{g_s(\mu)}{g(\mu)},$$

where

$$\dot g_s(\mu) = \sum_{k=1}^{\infty}\sum_{j=k+s}^{\infty} g_j\mu^k, \qquad \tilde g_s(\mu) = \int_0^{\infty} (1 - G(t+s))e^{\mu t}\,dt.$$

We define the transforms ${}_{jQ}x^{y,s}(\mu)$ and $P_x(\mu)$ analogously to (4).

Theorem 1. 1)

$${}_{jQ}x^{y,s}(\mu) = {}_{jP}x^{y,s}(\mu)\,g(\mu), \qquad j = 1, 2, 3.$$

2)

$${}_j\dot p_x^{y,s}(\mu\theta) = \theta^{m_2k_1-s}\,{}_j\dot p_x^{y,s}(\mu), \qquad j = 1, 2, 3,$$

where $\theta = e^{2\pi i/d}$, k_1 is a solution of the congruence

$$k_1m_1 \equiv x - y \pmod{d_1}$$

for $j = 2, 3$, and of the congruence

$$k_1m_1 \equiv x + y \pmod{d_1}$$

for $i = 1$; m_i are residues modulo d_i of possible values $\xi_k^{(i)}$.

3) There exist $\delta > 0, \gamma > 0$ such that, as $x \to \infty,\ y \to \infty$,

$$P_x(\mu) = D(\mu)\lambda_+^{-x}(g(\mu))\frac{W_{g(\mu)+}(1)}{W_{g(\mu)+}(\lambda_+(g(\mu)))}\left(1 + O(e^{-\gamma x})\right)$$

$$\left(\mu \in E_\delta \cap K_{\delta+}, \quad \dot D(\mu) = \frac{\mu}{1-\mu}, \quad \widetilde D(\mu) = -\frac{1}{\mu}\right),$$

$${}_2P_x^{y,s}(\mu) = T(\lambda_+(g(\mu)), \lambda_-(g(\mu)))\{1 + \Delta(\mu)O(e^{-\gamma x} + e^{-\gamma y})\}$$

$$(\mu \in E_\delta \cap K_{\delta+} \cap K_{\delta-}),$$

$${}_3P_x^{y,s}(\mu) = \begin{cases} -T(\lambda_+(g(\mu)), \lambda_+(g(\mu)))\{1 + \Delta(\mu)O(e^{-\gamma(x-y)})\}, \\ \qquad \text{for } x > y,\ \mu \in E_\delta \cap K_{\delta+}, \\ T(\lambda_-(g(\mu)), \lambda_-(g(\mu)))\{1 + \Delta(\mu)O(e^{\gamma(x-y)})\}, \\ \qquad \text{for } x < y,\ \mu \in E_\delta \cap K_\delta. \end{cases}$$

Here the estimates are uniform in μ and s. The absolute values of the functions $P_x(\mu)$, ${}_{jP}x^{y,s}(\mu)$ $(j = 2, 3)$ on the contours $|\mu| = \text{const}$, $\delta \leq |\arg\mu| < \pi/d$ in the lattice case and $\operatorname{Re}\mu = \text{const}$, $|\arg\mu| \geq \delta$, in the nonlattice case do not exceed their absolute values at the points of contact of these contours with the boundary, respectively, of one of the domains $K_{\delta+}, K_{\delta-}$; $\Delta(\mu) = \lambda_+(g(\mu)) - \lambda_-(g(\mu))$.

If y is fixed, then the following holds.

Theorem 2. For $\mu \in E_\delta \cap K_{\delta+}$ and $x \to \infty$,

$$ {}_1P_x^{y,s}(\mu) = \lambda_+^{-x}(g(\mu))\frac{g_s(\mu)}{g(\mu)}\frac{w_{g(\mu)+}(y)}{W_{g(\mu)+}(\lambda_+(g(\mu)))}\left(1 + O(e^{-\gamma x})\right), \qquad \text{if } y \geq 0, $$

$$ {}_1P_x^{y,s}(\mu) = \lambda_+^{-x}(g(\mu))\frac{g_s(\mu)}{g(\mu)}\frac{v_{g(\mu)+}(y)}{W_{g(\mu)+}(\lambda_+(g(\mu)))}\left(1 + O(e^{-\gamma x})\right), \qquad \text{if } y < 0, $$

where the estimates are uniform in μ, s (and for $y > 0$, also in y); $v_z(y)$ are the coefficients

for λ^y in the expansion of the function

$$ -\lambda\left[(\lambda - \lambda_-(z))(\lambda - \lambda_+(z))\,W_{z-}(\lambda)\right]^{-1}. \tag{5} $$

The values of ${}_1P_x^{y,s}(\mu)$ for $\mu \in E_\delta$ and $\mu \in K_{\delta+}$ are estimated in the same way as in Theorem 1.

The asymptotic analysis of the distributions themselves $P_{x,t}^{y,s}$, ${}_1p_{x,t}^{y,s}$, ${}^{y,s}_{jp_{x,t}}$, ${}^{y,s}_{jQ_{x,t}}$ is carried out by means of the saddle-point method. The results obtained make it possible, for example, to describe completely the large deviations for Kolmogorov-Smirnov statistics over the entire conceivable range of deviations. Here we give only one theorem, clarifying the physical meaning of the functions $W_{z\pm}(\lambda)$.

Theorem 3. Let $x/t = \tau$, $\lim_{t\to\infty}\tau = a \geqslant 0$, $m(\lambda) = \lambda^{-\tau}M^{-1}(f^{-1}(\lambda))$, let λ_τ be the point at which $\min_\lambda m(\lambda)$ is attained, and let $g_\tau = f^{-1}(\lambda_\tau)$. Then, if λ_a is located in the domain of regularity of the function $m(\lambda)$, then for $y \geqslant 0$ and $x \to \infty$

$$ {}_1\bar{P}_{x,t}^{y,s} = A\Phi(\lambda_a)\,g_s(M(g_a))\,w_{g_a+}(y)\,t^{-1/2}m^t(\lambda_\tau)\,\Xi\left(\frac{1}{t}, \tau - a\right), \tag{6} $$

where $A = \tau$ for $a = 0$, and $A = A(\lambda_a) > 0$ for $a > 0$; $\Xi\left(\frac{1}{t}, \tau - a\right)$ is an asymptotic expansion in powers of $1/t$ and $\tau - a$. The functions $A(\lambda_a)$, $\Phi(\lambda_a)$ are known and do not depend on x, t, y, s. The function $m^t(\lambda_\tau)$ can also be represented in the form $m^t(\lambda_a)e^{tH(\tau-a)}$, where $H(\tau - a)$ is the generalized Cramér series.

The value of ${}_1P_{x,t}^{y,s}$ for $y < 0$ is obtained if in formula (6) $w_{g_a+}(y)$ is replaced by $v_{g_a}(y)$, and $\Phi(\lambda_a)$ by the likewise known value $\Phi_1(\lambda_a)$.

Thus the functions $W_{z+}(\lambda)$ and (5) turn out to be the generating functions of the corresponding limiting conditional distributions. The functions $W_{z+}(\lambda)(\lambda-\lambda_+(z))$ and $W_{z-}(\lambda)(\lambda-\lambda_-(z))\lambda^{-1}$, which in a narrower domain effect the factorization of the function $1-zf(\lambda)$, can also be given a certain probabilistic meaning. If we put (cf. (3))

$$r_+(z,\lambda)=\sum_{n=1}^{\infty}\sum_{x=1}^{\infty}\mathbf{P}\{\bar{s}_{n-1}=0,\ s_n=x\}\,z^n\lambda^x,$$

$$r_-(z,\lambda)=\sum_{n=1}^{\infty}\sum_{x=-\infty}^{0}\mathbf{P}\{\bar{s}_n=0,\ \max(s_1,\dots,s_{n-1})<s_n,\ s_n=x\}\,(z^n\lambda^x),$$

then

$$(1-zf(\lambda))=(1-r_+(z,\lambda))(1-r_-(z,\lambda)).$$

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References Cited

[1] A. A. Borovkov, DAN, **143**, No. 3, 510 (1962).
[2] A. A. Borovkov, *Siberian Mathematical Journal*, **3**, 5, 645 (1962).
[3] G. Baxter, *Pacific J. Math.*, **8**, 4, 649 (1958).

Note: Figure translations are in progress. See original paper for figures.

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