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Abstract

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Classes $L_{(p_1,p_2,\dots,p_k)}(\Omega_m)$ and an Embedding Theorem for Abstract Set Functions

(Presented by Academician S. L. Sobolev on 12 VII 1962)

Let Ω_m be a bounded domain of the m-dimensional Euclidean space R^m $(m \leq n)$. We denote each point $\bar{x} \in R^m$ by $\bar{x} = (x_{s_1}, x_{s_2}, \dots, x_{s_k})$, where s_i are positive integers satisfying the condition $s_1 + s_2 + \cdots + s_k = m$; $\bar{x}_{s_1} = (x_1, x_2, \dots, x_{s_1})$, $\bar{x}_{s_2} = (x_{s_1+1}, x_{s_1+2}, \dots, x_{s_1+s_2})$, …, $\bar{x}_{s_j} = (x_{s_1+s_2+\cdots+s_{j-1}+1}, x_{s_1+s_2+\cdots+s_{j-1}+2}, \dots, x_{s_1+s_2+\cdots+s_{j-1}+s_j})$, ….

Next, denote by R^{s_j} the space of vectors $\bar{x}_{s_j}$,

$$\Omega_{s_j} = R^{s_j} \cap \Omega_m, \qquad |x-y| = r = r_n = \sqrt{r_{s_1}^2 + r_{s_2}^2 + \cdots + r_{s_k}^2},$$

$$r_{s_j} = \left[\sum_{i=s_1+s_2+\cdots+s_{j-1}+1}^{s_1+s_2+\cdots+s_{j-1}+s_j} (x_i - y_i)^2\right]^{1/2},$$

$$r_{n-(s_1+s_2+\cdots+s_{j-1})} = \left(r_{s_j}^2 + r_{s_{j+1}}^2 + \cdots + r_{s_k}^2\right)^{1/2}.$$

We shall consider the set of functions $f(\bar{x})$ defined in Ω_m. Each function $f(\bar{x})$ may be regarded as a function of the variable vectors $\bar{x}_{s_1}, \bar{x}_{s_2}, \dots, \bar{x}_{s_k}$. Under such a consideration, to almost every vector $\bar{x}_{s_i}$ there corresponds an element of an abstract space–the function $f(\bar{x}_{s_1}, \bar{x}_{s_2}, \dots, \bar{x}_{s_{i-1}}, \bar{x}_{s_i}^0, \bar{x}_{s_{i+1}}, \dots, \bar{x}_{s_k})$ of the variable vectors $\bar{x}_{s_1}, \bar{x}_{s_2}, \dots, \bar{x}_{s_{i-1}}, \bar{x}_{s_{i+1}}, \dots, \bar{x}_{s_k}$. Thus one may introduce for consideration the set of abstract functions $f(\bar{x}_{s_1}, \bar{x}_{s_2}, \dots, \bar{x}_{s_k})$ for which the expression

$$\left(\int_{\Omega_{s_k}}\left(\int_{\Omega_{s_{k-1}}}\cdots\left(\int_{\Omega_{s_2}}\left(\int_{\Omega_{s_1}}|f|^{p_1}\,d\bar{x}_{s_1}\right)^{p_2/p_1}d\bar{x}_{s_2}\right)^{p_3/p_2}\cdots d\bar{x}_{s_{k-1}}\right)^{p_k/p_{k-1}}d\bar{x}_{s_k}\right)^{1/p_k}$$

is bounded.

We denote the set of such functions by $L_{(p_1,p_2,\dots,p_k)}(\Omega_m)$. We introduce a norm in it by the equality

$$\|f\|_{L_{(p_1,p_2,\dots,p_k)}(\Omega_m)} =$$

$$= \left(\int_{\Omega_{s_k}} \left(\int_{\Omega_{s_{k-1}}} \cdots \left(\int_{\Omega_{s_2}} \left(\int_{\Omega_{s_1}} |f|^{p_1} \, d\bar{x}_{s_1} \right)^{p_2/p_1} d\bar{x}_{s_2} \right)^{p_3/p_2} \cdots d\bar{x}_{s_{k-1}} \right)^{p_k/p_{k-1}} d\bar{x}_{s_k} \right)^{1/p_k} .$$

Next, let Ω_s denote either a domain of an s-dimensional Euclidean space, or a smooth manifold of s dimensions in a Euclidean space of a larger number of dimensions. We assume the s-dimensional measure of Ω_s to be finite and different from zero, and set

$$U(\bar{x}) = \int_{\Omega_s} \tau(\bar{y}) \, r^{-\lambda} \, d\bar{y}.$$

For $s = n$, $U(\bar{x})$ was studied in the works of S. L. Sobolev (1,2), V. I. Kondrashov (4), V. P. Il' in (5), and L. V. Kantorovich (6); for $s < n$ and $m = n$, in the works of L. V. Kantorovich (6) and Kh. L. Smolitskii (7).

In this note we present results that generalize (in the case of a finite exponent) some results for $U(\bar{x})$ obtained in the works cited above. These results made it possible to generalize the embedding theorem of S. L. Sobolev (3) for abstract set functions, i.e., made it possible to formulate and prove an embedding theorem for the trace on arbitrary hyperplanes of dimension $s \le n$.

Theorem 1. If $\tau(\bar{y}) \in L_p(\Omega_s)$; $\lambda < \frac{s}{p'} + \sum_{i=1}^{k} \frac{s_i}{p_i}$, where s_i are positive rational numbers satisfying the condition $\sum_{i=1}^{k} s_i = n$, $s \le n$, and $1 < p \le p_1 \le p_2, p_3, \dots, p_k < \infty$, then $U(\bar{x}) \in L_{(p_1,p_2,\dots,p_k)}(\Omega_n)$ and, moreover, the inequality

$$\|U\|_{L_{(p_1,p_2,\dots,p_k)}(\Omega_n)} \le c \|\tau\|_{L_p(\Omega_s)}$$

holds.

Let the number ε be such that

$$\lambda = \frac{s}{p'} + \sum_{i=1}^{k} \frac{s_i}{p_i} - \varepsilon.$$

Put $\varepsilon = \varepsilon_1 + \varepsilon_2$; then

$$|U(\bar{x})| \le \int_{\Omega_s} r^{-s/p' + \varepsilon_1} \, |\tau(\bar{y})|^{p\left(\frac{1}{p} - \frac{1}{p_1}\right)} |\tau(\bar{y})|^{p/p_1} r^{-\left(\sum_{i=1}^{k} \frac{s_i}{p_i} - \varepsilon_2\right)} \, d\bar{y}. \qquad (1)$$

Applying Hölder' s inequality to three factors, after obvious transformations we obtain

$$|U(\bar{x})|^{p_1} \leq c_1 \|\tau\|_{L_p(\Omega_s)}^{p_1(1-p/p_1)} \int_{\Omega_s} |\tau(\bar{y})|^p r^{-\left(\sum_{i=1}^{k} \frac{s_i}{p_i} - \varepsilon_2\right)p_1} \, d\bar{y}. \tag{2}$$

Integrating (2) over Ω_{s_1}, changing the order of integration and simplifying, we shall have

$$\int_{\Omega_{s_1}} |U(\bar{x})|^{p_1} \, d\bar{x}_{s_1} \leq c_1 \|\tau\|_{L_p(\Omega_s)}^{p_1(1-p/p_1)} \int_{\Omega_s} |\tau(\bar{y})|^p r_{n-s_1}^{-\left(\sum_{i=2}^{k} \frac{s_i}{p_i} - \varepsilon_2\right)p_1} \, d\bar{y}. \tag{3}$$

Raising both sides of inequality (3) to the power p_2/p_1, integrating over Ω_{s_2}, applying the generalized Minkowski inequality ((^{9}), p. 179), and simplifying, we obtain

$$\int_{\Omega_{s_2}} \left(\int_{\Omega_{s_1}} |U(\bar{x})|^{p_1} \, d\bar{x}_{s_1} \right)^{p_2/p_1} d\bar{x}_{s_2} \leq c_2 \|\tau\|_{L_p(\Omega_s)}^{p_2(1-p/p_1)} \left(\int_{\Omega_s} |\tau(\bar{y})|^p r_{n-s_1-s_2}^{-\left(\sum_{3}^{k} \frac{s_i}{p_i} - \varepsilon_2\right)p_1} \, d\bar{y} \right)^{p_2/p_1}.$$

Continuing this process, at the k-th step we obtain

$$\|U\|_{L_{(p_1,p_2,\ldots,p_k)}(\Omega_n)}^{p_k} \leq c_k \|\tau\|_{L_p(\Omega_s)}^{p_k}.$$

Remark 1. For: 1) $s_1 = n, \ p_1 = \infty, \ s_2 = s_3 = \cdots = s_k = 0$; 2) $s_1 = n, \ p_1 < \infty$, we obtain assertions a) and b) of Theorem 2 of Kh. L. Smolitskii (7) and, in a somewhat different formulation, Theorem 1 of L. V. Kantorovich (6). For $s = n, \ s_1 < n, \ s_2 = s_3 = \cdots = s_k = 0, \ p = 2, \ p_1 < \infty, \ p_2 = \infty$, we obtain S. L. Sobolev' s result for an indefinite exponent (2); for $s = n, \ s_1 < n, \ s_3 = s_4 = \cdots = s_k = 0, \ p_1 < \infty, \ p_2 = \infty$, V. I. Kondrashov' s result.

Theorem 2. If $0 < p_1 \leq p_2, p_3, \ldots, p_k < \infty, \ k \leq s$, then

$$\mathbf{L}_{(p_1,p_2,\ldots,p_k)}(\Omega_s) \to \mathbf{L}_{p_1}(\Omega_s),$$

where $\to$ denotes embedding (see (8)).

Theorem 3. If $\tau(\bar{y}) \in \mathbf{L}_{(q',p_2,\ldots,p_k)}(\Omega_s)$;

$$\lambda < \frac{s}{q} + \frac{s_1}{p'} \sum_{i=2}^{k} \frac{s_i}{p_i},$$

where s_i are positive integral rational numbers satisfying the condition

$$\sum_{i=1}^{k} s_i = n, \qquad k \leq s$$

and

$$p \le q \le \max(q, p') \le p_2, p_3, \dots, p_k < \infty,$$

then $U(\bar{x}) \in \mathbf{L}_{p'}(\Omega_n)$, and, moreover, the inequality

$$\|U\|_{L_{p'}(\Omega_n)} \le c\|\tau\|_{L_{(q', p_2, \dots, p_k)}(\Omega_s)}. \tag{4}$$

holds.

Remark 2. For $s_2 = s_3 = \dots = s_k = 0$, from Theorem 3 we obtain the assertion of S. L. Sobolev ((3), p. 307).

Theorem 3 makes it possible to generalize S. L. Sobolev' s embedding theorem to abstract functions of sets. In order to formulate and prove the result obtained, let us consider the class $\Phi_{(p_1, p_2, \dots, p_k)}(X, \Omega \cap S_s)$ of abstract additive functions of sets $\varphi(I)$ ($I \in \mathcal{E}_s$, where $\mathcal{E}_s$ is the set of all Lebesgue-measurable subsets of $\Omega \cap S_s$) with values in a Banach space X, for which the norm introduced by the equality

$$\|\varphi\|_{\Phi_{(p_1, p_2, \dots, p_k)}(X, \Omega \cap S_s)} = \sup_{\tau} \frac{\left\| \int_S \tau(x)\, d_x \varphi(I) \right\|_X}{\|\tau\|_{L_{(p_1', p_2, \dots, p_k)}(\Omega \cap S_s)}} \tag{5}$$

is bounded. As is not difficult to see, from $\Phi_{(p_1, p_2, \dots, p_k)}$, as a special case one obtains the known space Φ_{p_1} of S. L. Sobolev.

Theorem 4. If $\varphi(E) \in \psi_p^{(l)}(X, \Omega)$,

$$n - l < \frac{s}{q} + \frac{s_1}{p'} + \sum_{i=2}^{k} \frac{s_i}{p_i},$$

where s_i are positive integral rational numbers satisfying the condition

$$\sum_{i=1}^{k} s_i = n, \qquad k \le s \le n,$$

and

$$p \le q \le \max(q, p') \le p_2, p_3, \dots, p_k < \infty,$$

then φ is defined on all smooth manifolds $S_s \cap \Omega$ of s dimensions and represents a function of sets $\tilde{\varphi}(I)$ belonging to the class $\Phi_{(q, p_2, \dots, p_k)}(X, \Omega \cap S_s)$. Moreover, the inequality

$$\|\tilde{\varphi}\|_{\Phi_{(q,p_2,\dots,p_k)}(X,\Omega\cap S_s)} \le c\|\varphi\|_{\psi_p^{(l)}(X,\Omega)}; \tag{6}$$

holds; $c = \text{const}$ does not depend on φ.

Consider the integral identity

$$\tilde{\varphi}_h(I) = \int_\Omega K(I,\bar{y})\, d_y \varphi_h(E) + \sum_{|\bar{\alpha}|=l} \int_\Omega K_{\bar{\alpha}}(I,\bar{y})\, d_y D^{\bar{\alpha}} \varphi_h(E),$$

where

$$\widetilde{\varphi}_h(I) = \int_I \overline{\varphi}(\overline{y})\, d\overline{y}; \qquad \varphi_h(E) = \int_E \overline{\varphi}(\overline{x})\, d\overline{x}; \qquad D^{\overline{\alpha}}\varphi_h(E) = \int_E D^{\overline{\alpha}}\varphi(\overline{y})\, d\overline{y};$$

$$K(I,\overline{y}) = \int_I \sum_{|\overline{\alpha}|\le l-1} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \xi_{\overline{\alpha}}(\overline{y})\, d\overline{x}; \qquad K_{\overline{\alpha}}(I,\overline{y}) = \int_I \frac{\omega_{\overline{\alpha}}(\overline{x},\overline{y})}{r^{n-l}}\, d\overline{x}.$$

Let us estimate the expression

$$\|\widetilde{\varphi}\|_{\Phi_{(q,p_2,\dots,p_k)}(X,\Omega\cap S_s)} \le \sup_\tau \frac{\left\| \int \tau(\overline{x})\, d_x \left[\int_\Omega K(I,\overline{y})\, d_y \varphi(E) \right] \right\|_X}{\|\tau\|_{L_{(q',p_2,\dots,p_k)}(\Omega\cap S_s)}} +$$

$$+ \sum_{|\overline{\alpha}|=l} \sup_\tau \frac{\left\| \int \tau(\overline{x})\, d_x \left[\int_\Omega K_{\overline{\alpha}}(I,\overline{y})\, d_y D^{\alpha}\varphi_h(E) \right] \right\|_X}{\|\tau\|_{L_{(q',p_2,\dots,p_k)}(\Omega\cap S_s)}}.$$

Using the definition of the integral (3) and making the necessary transformations, we obtain

$$\|\widetilde{\varphi}\|_{\Phi_{(q,p_2,\dots,p_k)}(X,\Omega\cap S_s)} \le$$

$$\le \sup_\tau \frac{\left\| \int_\Omega \left[\int_I \tau(\overline{x})\, d_x K(I,\overline{y}) \right]\, d_y \varphi_h(E) \right\|_X}{\left\| \int \tau(\overline{x})\, d_{xK}(I,\overline{y}) \right\|_{L_{p'}(\Omega)}} \frac{\left\| \int \tau(\overline{x})\, d_{xK}(I,\overline{y}) \right\|_{L_{p'}(\Omega)}}{\|\tau\|_{L_{(q',p_2,\dots,p_k)}(\Omega\cap S_s)}} +$$

$$+ \sum_{|\overline{\alpha}|=l} \sup_\tau \frac{\left\| \int_\Omega \left[\int_I \tau(\overline{x})\, d_x K_{\overline{\alpha}}(I,\overline{y}) \right]\, d_y D^{\alpha}\varphi_h(E) \right\|_X}{\left\| \int \tau(\overline{x})\, d_x K_{\overline{\alpha}}(I,\overline{y}) \right\|_{L_{p'}(\Omega)}} \frac{\left\| \int \tau(\overline{x})\, d_x K_{\overline{\alpha}}(I,\overline{y}) \right\|_{L_{p'}(\Omega)}}{\|\tau\|_{L_{(q',p_2,\dots,p_k)}(\Omega\cap S_s)}}. \tag{7}$$

On the basis of Theorem 3 and simple estimates we have

$$\left\|\int_I \tau(\overline{x})\, d_x K_{\overline{\alpha}}(I,\overline{y})\right\|_{L_{p'}(\Omega)} \leq c_1 \|\tau\|_{L_{(q',p_2,\dots,p_k)}(\Omega\cap S_s)}; \tag{8}$$

$$\left\|\int_I \tau(\overline{x})\, d_{xK}(I,\overline{y})\right\|_{L_{p'}(\Omega)} \leq c_2 \|\tau\|_{L_{(q',p_2,\dots,p_k)}(\Omega\cap S_s)}. \tag{9}$$

From (7), (8), and (9) we obtain

$$\|\widetilde{\varphi}_h\|_{\Phi_{(q,p_2,\dots,p_k)}(X,\Omega\cap S_s)} \leq c\|\varphi_h\|_{\psi_p^{(s)}(X,\Omega)}. \tag{10}$$

Passing to the limit in equality (10) as $h \to 0$, we obtain (6).

Remark 3. For $s_1 = n,\ s_2 = s_3 = \cdots = s_k = 0$, we obtain the theorem of S. L. Sobolev ([3], p. 321).

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