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Abstract

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SEMIGROUPS WITH A DEDEKIND STRUCTURE OF SUBSEMIGROUPS

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By $\Sigma'(\Gamma)$, as in $\varphi(^{1-4})$, we denote the set of all subsemigroups of a semigroup Γ, partially ordered by inclusion, regarding also the empty set as a subsemigroup. $\Sigma'(\Gamma)$ is a complete lattice.

In the present note we describe the structure of an arbitrary semigroup Γ for which $\Sigma'(\Gamma)$ is a Dedekind lattice. The main result of the note, in a certain sense, reduces this description to the description of periodic modular groups*. The structure of modular groups, as is known, has been almost completely studied (see $(^5)$), although a complete description of periodic modular groups apparently still does not exist (but, for example, the structure of locally finite modular groups has been determined).

We shall often use the following property of lattices, equivalent to Dedekindness (we formulate it at once for the lattice of subsemigroups of a semigroup): if A, B, C are arbitrary subsemigroups such that $A \subseteq C$, $A \cap B = C \cap B$, and $\{A, B\} = \{C, B\}$, then $A = C$.

We use the same notation as in our previous papers (see, for example, $(^4)$). By K_e, in particular, we denote the set of all elements of the given periodic semigroup some power of which is equal to the idempotent e of this semigroup. As is known (see $(^6)$), an arbitrary periodic semigroup is the set-theoretic sum of its pairwise disjoint classes K_e, which are not always subsemigroups. In the paper we shall also make essential use of the concept of a strong bond of semigroups introduced by us in $(^2)$.

First of all let us recall that, as was noted in $(^2)$, a semigroup with a Dedekind structure of subsemigroups is a periodic semigroup.

Lemma 1. *Let e be an arbitrary idempotent of a semigroup Γ with Dedekind structure $\Sigma'(\Gamma)$, and let x be an arbitrary element of Γ not belonging to K_e. Then the subsemigroup generated by the elements e and x coincides with the set-theoretic sum of the cyclic subsemigroups generated by these elements.*

Proof. Consider the element ex. Suppose that $ex \neq e$, and then show that $ex \in \{x\}$. We have $\{\{e, ex\}, x\} = \{e, x\}$, and by assumption $\{e\} \subset \{e, ex\}$ (proper inclusion). It follows from the condition that $\{e\} \cap \{x\} = \emptyset$. Taking this and the preceding into account, we obtain that the equality

$$\{e, ex\} \cap \{x\} = \emptyset$$

is impossible, since otherwise we would arrive at a contradiction: the above condition of Dedekindness would not be satisfied.

Denote $\{e, ex\} \cap \{x\} = P$. Thus, we have established that $P \neq \emptyset$. The following two equalities are evident:

$$\{e, P\} \cap \{x\} = \{e, ex\} \cap \{x\},$$

$$\{\{e, P\}, x\} = \{\{e, ex\}, x\}.$$

From these equalities and from the inclusion $\{e, P\} \subseteq \{e, ex\}$ it follows, by virtue of the Dedekindness of $\Sigma'(\Gamma)$, that $\{e, P\} = \{e, ex\}$, whence, in turn, it follows that $ex \in \{e, P\}$.

* Following (5), by a **modular** group we mean a group with a Dedekind structure of subgroups.

It is easy to see that e is a left identity in the semigroup $\{e, ex\}$; therefore $\{e, P\} = \{e\} \cup P \cup Pe$. Thus we have obtained $ex \in \{e\} \cup P \cup Pe$. Since, by assumption, $ex \neq e$, it follows that $ex \in P \cup Pe$. If $ex \in P$, then all the more $ex \in \{x\}$, as was required to prove. Let $ex \in Pe$, i.e. $ex = x^m e$, where $x^m \in P$, and m is some natural number. Multiplying both sides of the last equality on the right by e, we obtain $exe = ex$. We shall show that for any natural number n one also has $ex^n e = ex^n$. Indeed, suppose the equality $ex^{n-1}e = ex^{n-1}$ has already been proved. Then

$$ex^n e = ex^{n-1}xe = ex^{n-1}exe = ex^{n-1}ex = ex^{n-1}x = ex^n.$$

In particular, $ex^m e = ex^m$. But $ex^m = x^m$, since $x^m \in P$, and e is a left identity in $\{e, ex\}$. Therefore the equality $x^m e = x^m$ is valid. But $x^m e = ex$. Consequently, $ex \in \{x\}$, as was required to prove.

Quite analogously it is proved that from the assumption $xe \neq e$ it follows that $xe \in \{x\}$.

The assertion of the lemma is an immediate consequence of the properties obtained for the elements ex and xe.

Lemma 2. *For arbitrary classes K_e and K_i of a semigroup Γ with Dedekind structure $\Sigma'(\Gamma)$, the inclusion*

$$K_{eK}i \subseteq K_e \cup K_i$$

holds.

Proof. Let x and y be arbitrary elements of the classes K_e and K_i, respectively. Suppose that $xy \notin K_e \cup K_i$. Then $xy \in K_f$, where f is an idempotent different from e and from i. By Lemma 1 we have $\{x, f\} = \{x\} \cup \{f\}$. In view of the equality $\{y\} \cap \{f\} = \emptyset$, we obtain from this

$$\{x, f\} \cap \{y\} = \{x\} \cap \{y\}.$$

Moreover, from the condition $f \in \{x, y\}$ it follows that

$$\{\{x, f\}, y\} = \{x, y\}.$$

From the Dedekind property of $\Sigma'(\Gamma)$ it then follows that $\{x, f\} = \{x\}$, whence $f \in \{x\}$, a contradiction. The contradiction obtained proves the lemma.

For $e = i$, from Lemma 2 we obtain the following important

Corollary. *If Γ is a semigroup with Dedekind structure $\Sigma'(\Gamma)$, then every class K_e of this semigroup is its subsemigroup.*

Lemma 3. *If Γ is a semigroup with Dedekind structure $\Sigma'(\Gamma)$, then Γ is a strong semilattice of semigroups, each of which contains one idempotent.*

Proof. By the corollary to Lemma 2, every class K_e of the semigroup Γ is its subsemigroup. We shall show that Γ is a strong semilattice of these subsemigroups. The case when Γ has one idempotent is trivial. Let e, i be two arbitrary distinct idempotents of the semigroup Γ; let x, y be arbitrary elements of K_e and K_i, respectively. By Lemma 1 we have $\{e, i\} = \{e\} \cup \{i\}$, i.e. $ei = e$ or $ei = i$.

Let, for example, $ei = e$. We shall show that then $xy \in K_e$. Suppose the contrary: $xy \notin K_e$. Then $xy \in K_i$ by Lemma 2. By the condition $y^m = i$ for some natural m, and, obviously, one may assume $m > 1$. Since $y \in K_i$, on the basis of the corollary to Lemma 2 we have

$$xi = xy^m = xy \cdot y^{m-1} \in K_i.$$

But by Lemma 1, $xi \in \{x\} \cup \{i\}$. Consequently, $xi = i$. By the condition $x^n = e$ for some natural n. Hence $ei = x^n i = i$, which contradicts the equality $ei = e$.

The contradiction obtained shows that $xy \in K_e$. By the arbitrariness of the elements x and y it follows that $K_{eK}i \subseteq K_e$. Analogously, from $ei = i$ it follows that $K_{eK}i \subseteq K_i$. This means that Γ is a semilattice (see ([7], [6])) of its subsemigroups K_e, where e runs through the set of all idempotents of Γ. To prove that Γ will be a strong semilattice of these subsemigroups (components of the semilattice), it remains to show that the composite of two arbitrary subsemigroups from different components (i.e. the subsemigroup generated by them)

coincides with their set-theoretic sum. This property, obviously, it is enough to prove for arbitrary cyclic subsemigroups from different components of connection.

Thus, let again x, y be arbitrary elements from different components K_e and K_i, respectively. Let $xy \in K_e$. Since $\{xy, x\} \subseteq K_e$, by the consequence of Lemma 2, we have

$$\{xy, x\} \cap \{y\} = \{x\} \cap \{y\} = \emptyset.$$

Moreover,

$$\{\{xy, x\}, y\} = \{x, y\}.$$

From the Dedekindness of $\Sigma'(\Gamma)$ we then obtain

$$\{xy, x\} = \{x\},$$

whence $xy \in \{x\}$. Similarly, from $xy \in K_i$ we would obtain $xy \in \{y\}$. Thus always $xy \in \{x\} \cup \{y\}$. Hence it follows that

$$\{x, y\} = \{x\} \cup \{y\}.$$

This completes the proof of the lemma.

Since a subsemigroup of a semigroup with a Dedekind structure of subsemigroups itself has this property, taking Lemma 3 into account, it remains now to study the structure of a semigroup with one idempotent whose structure of subsemigroups is Dedekind. Let Γ be such a semigroup. Like any periodic semigroup with one idempotent, Γ is an extension ([8], [9]) of a periodic group G by a nilsemigroup (see ([1], [4])). G is a modular group.

Since a homomorphic image of a Dedekind structure will be a Dedekind structure, according to Lemma 3.4 of paper ([4]) the factor semigroup $\Gamma - G$ is a nilsemigroup with a Dedekind structure of subsemigroups. Then by Theorem 2.15 of paper ([9]) the composite of any two of its subsemigroups coincides with their set-theoretic sum. From this condition, in particular, it follows that the semigroup $\Gamma - G$ will be nilpotent of class ≤ 5 ([9]).

Let now Γ be an extension of a periodic modular group G by a nilpotent semigroup with the condition just indicated. We note that this condition means that for any subsemigroups $A, B \subseteq \Gamma$ one has

$$\{A, B\} \subseteq A \cup B \cup G.$$

The idempotent e of the semigroup Γ, which is the identity of the group G, belongs to every subsemigroup of Γ and commutes with every element of Γ. From these properties, taking into account that G is an ideal of Γ, we obtain, as is easy to see, that for any subsemigroups $A, B \subseteq \Gamma$ the inclusion

$$e\{A, B\} \subseteq \{A \cap G,\ B \cap G\}$$

holds.

We note, furthermore, that $\Sigma'(G)$ is Dedekind, since every subsemigroup in G is a subgroup.

We shall show that $\Sigma'(\Gamma)$ is Dedekind. Let A, B, C be arbitrary subsemigroups of Γ such that $A \subseteq C$. Denote

$$A \cap G = A_1, \qquad B \cap G = B_1, \qquad C \cap G = C_1.$$

Obviously, $A_1 \subseteq C_1$. Let x be an arbitrary element of $\{A, B\} \cap C$. We have $x \in C$ and

$$x \in \{A, B\} \subseteq A \cup B \cup G.$$

If $x \notin G$, then $x \in A \cup B$, and then $x \in \{A, B \cap C\}$. Let $x \in G$. In this case $ex = x$. But

$$ex \in e\{A, B\} \subseteq \{A \cap G,\ B \cap G\} = \{A_1, B_1\},$$

so that $x \in \{A_1, B_1\} \cap C_1$. $\Sigma'(G)$ is Dedekind; therefore

$$\{A_1, B_1\} \cap C_1 = \{A_1,\ B_1 \cap C_1\}.$$

But

$$\{A_1,\ B_1 \cap C_1\} \subseteq \{A,\ B \cap C\}.$$

Thus, in all cases $x \in \{A, B \cap C\}$, i.e.

$$\{A, B\} \cap C \subseteq \{A, B \cap C\}.$$

Since the reverse inclusion always holds, we have

$$\{A, B\} \cap C = \{A, B \cap C\},$$

which proves the Dedekindness of $\Sigma'(\Gamma)$.

Thus, the following has been proved.

Lemma 4. If Γ is a semigroup with one idempotent, then $\Sigma'(\Gamma)$ is Dedekind if and only if Γ is an extension of a periodic modular group by means of a nilpotent semigroup in which the composite of any two subsemigroups coincides with their set-theoretic sum.

From Lemmas 3 and 4 it follows that an arbitrary semigroup with a Dedekind structure of subsemigroups is a strong semilattice of semigroups having the structure indicated in Lemma 4. Conversely, let the semigroup Γ be a strong semilattice of semigroups having the structure indicated in Lemma 4. From this lemma it follows that the structure of subsemigroups of each of the components of connection will be Dedekind. Then $\Sigma'(\Gamma)$ will also be Dedekind, which follows from

of the following Lemma 5 and from the fact that the direct product of an arbitrary set of Dedekind lattices is again a Dedekind lattice.

Lemma 5. *If a semigroup Γ is a strong semilattice of semigroups A_α (where α ranges over some set of indices $\mathfrak{A}$), then $\Sigma'(\Gamma)$ is isomorphic to the direct product of the lattices $\Sigma'(A_\alpha)$ over all $\alpha \in \mathfrak{A}$.*

We omit the proof of Lemma 5 here.

Thus, we have proved the following

Theorem. $\Sigma'(\Gamma)$ *is Dedekind if and only if the semigroup* Γ *is a strong semilattice of semigroups, each of which is an extension of a modular periodic group by means of a nilpotent semigroup in which the composite of any two subsemigroups coincides with their set-theoretic sum.*

Using this theorem, it is easy, in particular, to obtain the description of semigroups with a distributive lattice of subsemigroups given by us in ([2], [4]) (see also ([10]), where an analogous description was obtained independently*).

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REFERENCES

[1] L. N. Shevrin, DAN, **138**, No. 1, 73 (1961).
[2] L. N. Shevrin, DAN, **138**, No. 4, 796 (1961).
[3] L. N. Shevrin, DAN, **144**, No. 1, 72 (1962).
[4] L. N. Shevrin, Sibirsk. matem. zhurn., **3**, No. 3, 477 (1962).
[5] M. Suzuki, *Structure of a Group and the Structure of Its Lattice of Subgroups*, Moscow, 1960.
[6] E. S. Lyapin, *Semigroups*, Moscow, 1960.
[7] A. H. Clifford, Proc. Am. Math. Soc., **5**, No. 3, 499 (1954).
[8] A. H. Clifford, Trans. Am. Math. Soc., **68**, No. 2, 165 (1950).
[9] L. N. Shevrin, Matem. sborn., **53** (95), No. 3, 367 (1961).
[10] M. Ego, C. R., **252**, No. 17, 2490 (1961).
[11] M. Ego, C. R., **254**, No. 10, 1723 (1962).

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* *Note added in proof.* After the present paper had already been sent to press, the author became aware of the paper ([11]), in which semigroups with a Dedekind lattice of subsemigroups are described.

Note: Figure translations are in progress. See original paper for figures.

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