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Abstract

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PERIODIC SUBGROUPS OF ALGEBRAIC GROUPS

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Periodic subgroups of linear groups have been studied in a number of works by D. A. Suprunenko and his students. Thus, in ([1], [2]) it was proved, respectively, that Sylow p-subgroups and maximal irreducible locally nilpotent periodic subgroups of the full linear group $L_n(P)$ over an algebraically closed field P are conjugate, and in ([3]) the conjugacy of maximal periodic subgroups in a maximal solvable subgroup $M \subset L_n(P)$ was proved. However, the method of proof in ([1]-[3]) makes essential use of the maximality of the groups under consideration and cannot be applied to broader classes of groups, so that the problem of conjugacy of Sylow p-subgroups in arbitrary linear groups remained little studied.

In the present article the problem of conjugacy of Sylow p-subgroups in linear groups is, in a certain sense, solved completely, namely: in every algebraic linear group the Sylow p-subgroups are conjugate. At the same time there exist linear groups (see the example below) having an infinite number of pairwise nonisomorphic (not to mention conjugate) Sylow p-subgroups. This gives a negative answer to D. A. Suprunenko' s question as to whether the number of isomorphism classes of Sylow p-subgroups must be finite. In addition, this article proves the conjugacy of maximal periodic subgroups in a solvable algebraic group (in an arbitrary solvable linear group even Sylow p-subgroups need not be conjugate) and completely establishes the structure of solvable algebraic groups over fields of arbitrary characteristic (for the case of a field of characteristic zero this was done in ([8])). The proofs rely on results of A. Borel and J. Serre ([6]), A. I. Mal' tsev ([7]), and the author ([8]). It should be noted that the use of topological methods makes it possible to carry out the proofs almost without computations.

In what follows the field P is assumed to be algebraically closed; all topological notions refer to the Zariski topology; by an algebraic group we always mean a linear algebraic group; p, q are prime numbers.

A group H is called **complete** if the equation $x^m = h$ (m an arbitrary integer) is solvable in H for every $h \in H$.

Lemma 1. *If a group G has a solvable normal divisor R of finite index, whose elements of finite order form a subgroup R_0, and the factor group R/R_0 is a complete nilpotent torsion-free group, then in G the maximal periodic subgroups are conjugate.*

The proof is based on the well-known lemma on the mean in abstract groups ([7], Lemma 6).

A group of matrices conjugate to a subgroup of the group of diagonal matrices will be called **diagonalizable**. From Proposition 7.1 in ([5]) it follows:

Lemma 2. *A connected abelian algebraic diagonalizable group is complete.*

A maximal connected diagonalizable subgroup T of an algebraic group G is called a **maximal torus** in G. As shown by A. Borel ([5], Corollary 16.6), in every algebraic linear group maximal tori are internally conjugate.

Of greatest importance for the proof of the main theorem is

Theorem 1. *A completely reducible locally nilpotent periodic subgroup Q of an algebraic group G belongs to the normalizer $N(T)$ of some maximal torus T in G.*

The **proof** of Theorem 1 is carried out by a method which is a refinement of the method of A. Borel and J.-P. Serre ([6]); see also ([9]) or ([10]), Chapter 20. In the case where the field P has characteristic zero, Theorem 1 becomes the following theorem, which is undoubtedly of independent interest. By the algebraic algebra of a Lie algebra is meant the Lie algebra of an algebraic group (see ([12])).

Theorem 2. *Let Φ be a locally nilpotent periodic group of automorphisms of an algebraic Lie algebra R; then there exists a maximal commutative subalgebra of semisimple (diagonalizable) endomorphisms in R, invariant with respect to Φ.*

In ([5]), Corollary 16.8, the following assertion is proved:

Theorem 3. *The connected component $N_0(T)$ of the normalizer $N(T)$ of a maximal torus T in an algebraic group G is nilpotent.*

It is proved in an obvious way.

Lemma 3. *In a locally finite group possessing a normal divisor of finite index with an invariant Sylow p-subgroup (p a prime), all Sylow p-subgroups are conjugate.*

Corollary. *In a periodic linear group Γ whose element orders are not divisible by the characteristic of the field P, the Sylow p-subgroups are conjugate.*

Indeed, by Schur' s theorem ([11]), Γ possesses an abelian normal divisor of finite index.

From Theorems 1, 3, Lemmas 1, 2, 3 of the present paper, Theorem 16.5 and Corollary 16.6 from ([5]), there follows the main

Theorem 4. *In every algebraic linear group the Sylow p-subgroups are conjugate.*

$L_n(P)$ is certainly an algebraic group; therefore the following is valid.

Corollary (1). *In $L_n(P)$ the Sylow p-subgroups are conjugate.*

This same result can be derived directly from Lemmas 1 and 3, if one takes into account that for $p = q$ (q is the characteristic of the field P) the Sylow p-subgroups are conjugate to the special triangular group ([7]), while for $p \neq q$ every p-group is monomial.

Example. Let $A_1, A_2, \dots, A_k, \dots$ be an infinite sequence of arbitrary p-groups from $L_n(P)$. Consider the free product

$$A = \prod_i^{*} A_i.$$

It is easy to show (see ([4]), p. 349) that the A_i ($i = 1, 2, \dots$) are Sylow p-subgroups in A, and are obviously pairwise nonconjugate. At the same time, by Theorem 1 from ([13]), the group A is isomorphically representable by matrices of degree $n+1$, i.e. one may assume that $A \subset L_{n+1}(P)$; consequently, A is a linear group having infinitely many pairwise nonconjugate Sylow p-subgroups. It is easy to see that the A_i may even be pairwise nonisomorphic. Thus, in an arbitrary linear group Γ there may exist infinitely many classes of isomorphic Sylow p-subgroups, whereas in the group $\overline{\Gamma}$ ($\overline{\Gamma}$ is the closure of Γ in the Zariski topology) the Sylow p-subgroups are conjugate by Theorem 4.

We turn to the study of solvable algebraic groups.

Theorem 5. *If the index $m = \Gamma : \Gamma_0$ of the connected component Γ_0 of a solvable algebraic group Γ is not divisible by the characteristic q, then $\Gamma = D \cdot \Gamma_u$, $D \cap \Gamma_u = (e)$, where D is a maximal d-subgroup of Γ, and Γ_u is the unipotent part of Γ. All maximal d-subgroups (generalized tori) of the group Γ are conjugate in Γ.*

The **proof** of Theorem 5 is analogous to the proof of Theorem A in ([8]) and uses the following lemma, which follows from the well-known Schur–Zassenhaus theorem (see ([4]), p. 386, and also ([14])).

Lemma 4. If the unipotent part G_u of a linear group G is a subgroup of finite index in G, then $G = K \cdot G_u$, $K \cap G_u = (e)$, and all K possessing these properties are conjugate in G.

The following theorem clarifies the structure of solvable algebraic groups over fields of arbitrary characteristic.

Theorem 6. Let Γ be a solvable algebraic group and

$$m = \Gamma : \Gamma_0 = \pi \cdot q^{\alpha},$$

where $(\pi, q) = 1$, q is the characteristic of the field P, and Γ_0 is the connected component of Γ. Then Γ has subgroups of index q^{α}, conjugate in Γ. If H is one of them, then $H = D \cdot H_u$, $D \cap H_u = (e)$, where D is a maximal d-subgroup of the group Γ. All maximal d-subgroups of the group Γ are conjugate in Γ.

Proof for the case of a field of characteristic zero was given in ([8]). If $q > 0$, then the proof follows from Theorem 5 and the well-known theorem of P. Hall ([15]) on finite solvable groups, according to which in the group $\Gamma^* = \Gamma/\Gamma_0$ there are Hall π-subgroups, and all of them are conjugate in Γ^*. The last assertion of Theorem 6 follows from the fact that every d-subgroup is contained (by P. Hall' s theorem) in some subgroup H of index q^α in Γ.

According to Theorem 1 of ([7]), a completely reducible solvable group G has a diagonalizable normal divisor of finite index; consequently, the connected component of the group G is diagonalizable. Therefore, from Lemmas 1 and 2 it follows that

Lemma 5. In a solvable algebraic completely reducible group, the maximal periodic subgroups are conjugate.

Theorem 7. In every solvable algebraic group, the maximal periodic subgroups are conjugate.

Proof is based on Lemmas 1, 2, 5, Theorem 6 of the present paper, and Theorem 12.3 of ([5]).

A maximal solvable subgroup $M \subset L_n(P)$ is algebraic; therefore, from Theorem 7 the results of ([3]) follow.

Corollary 1. In a maximal solvable subgroup of $L_n(P)$, the maximal periodic subgroups are conjugate.

Corollary 2. In $L_n(P)$, the number of conjugacy classes of maximal solvable periodic subgroups is finite.

In conclusion, we note that Theorem 4 of the present paper is also valid for connected linear groups of Lie.

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Note: Figure translations are in progress. See original paper for figures.