THEORY OF ELASTICITY

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Abstract

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ON THE VIBRATIONS OF ELASTIC RODS

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In the problem of vibrations of elastic rods (see, for example, ([1])), the hypothesis of plane sections often leads to substantial errors. V. Z. Vlasov (see ([2]), Ch. X), neglecting plane deformations of transverse sections, refined the equilibrium equations of an elastic rod by approximately taking into account the warping of these sections. In this paper we indicate a method for reducing the three-dimensional problem of the theory of elasticity to a one-dimensional one, which makes it possible, without introducing any simplifying hypotheses, to solve static and dynamic problems of the theory of elastic rods with errors of arbitrarily high order of smallness in comparison with the smallness of the ratio r/l, where l is the length of the rod and r is the maximum distance of the particles of the rod from its axis. This method, adjoining the methods developed by V. Z. Vlasov, very simply reduces the general problem to an infinite system of one-dimensional linear differential equations possessing a structure convenient for investigation. Applying to this system the known method of asymptotic integration of differential equations containing a small parameter, one can construct, for the required elastic displacements, asymptotic representations convenient for carrying out practical calculations.

Taking the axis x of the rectangular coordinate system x, y, z to coincide with the axis of the rod, let us introduce into consideration the vectors

$$\mathbf{F}_{kl}(x,t) = \lim_{h\to 0} \frac{1}{h}\left(\iint_s \mathbf{p}\, y^k z^l\, ds + \iiint_v \mathbf{P}\, y^k z^l\, dv\right), \qquad (1)$$

where $\mathbf{p}$ and $\mathbf{P}$ are vectors determining the surface and volume forces acting on the rod; s is the lateral surface cut out by plane sections of the rod normal to the axis and having coordinates $x-h/2$ and $x+h/2$; v is the region bounded by these plane sections and the lateral surface s. In formula (1), in accordance with the three-dimensional equations of vibrations of an elastic body, the projections of the vectors $\mathbf{p}$ and $\mathbf{P}$ on the coordinate axes x, y, z may be expressed in terms of the normal stresses $\sigma_x, \sigma_y, \sigma_z$, the shear stresses $\tau_{xy}, \tau_{xz}, \tau_{yz}$, and the volume inertia forces $-\rho\dfrac{\partial^2 u_x}{\partial t^2}, -\rho\dfrac{\partial^2 u_y}{\partial t^2}, -\rho\dfrac{\partial^2 u_z}{\partial t^2}$. By means of the Gauss–Ostrogradsky formulas, the integration over the surface s can then be replaced by integration over the region v and over the plane sections bounding this region. Passing after this to the limit, we obtain, for the projections X_{kl}, Y_{kl}, Z_{kl} of the vector $\mathbf{F}_{kl}$ on the coordinate axes x, y, z, the formulas

$$X_{kl} = \iint_S \left(k\tau_{xy}z + l\tau_{xz}y + \rho \frac{\partial^2 u_x}{\partial t^2} yz \right) y^{k-1} z^{l-1}\, dS - \frac{\partial}{\partial x} \left(\iint_S \sigma_x y^k z^l \, dS \right),$$

$$Y_{kl} = \iint_S \left(k\sigma_y z + l\tau_{yz}y + \rho \frac{\partial^2 u_y}{\partial t^2} yz \right) y^{k-1} z^{l-1}\, dS - \frac{\partial}{\partial x} \left(\iint_S \tau_{xy} y^k z^l \, dS \right),$$

$$Z_{kl} = \iint_S \left(k\tau_{yz}z + l\sigma_z y + \rho \frac{\partial^2 u_z}{\partial t^2} yz \right) y^{k-1} z^{l-1}\, dS - \frac{\partial}{\partial x} \left(\iint_S \tau_{xz} y^k z^l \, dS \right), \tag{2}$$

where S is the transverse section of the rod with coordinate x.

We shall seek the elastic displacements u_x, u_y, u_z in the form of double power series

$$u_x = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} u_{ij}(x,t) y^i z^j, \qquad u_y = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} v_{ij}(x,t) y^i z^j,$$

$$u_z = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_{ij}(x,t) y^i z^j. \tag{3}$$

Expressing the stresses entering formulas (2) in terms of the elastic displacements u_x, u_y, u_z and substituting the series (3) into (2), we obtain for the functions u_{ij}, v_{ij}, w_{ij} an infinite system of equations

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \left[\frac{\partial}{\partial x} \left(N_{i+k,j+l} \frac{\partial u_{ij}}{\partial x} + iL_{i+k-1,j+l} v_{ij} + jL_{i+k,j+l-1} w_{ij} \right) \right.$$

$$\left. -kM_{i+k-1,j+l} \frac{\partial v_{ij}}{\partial x} - lM_{i+k,j+l-1} \frac{\partial w_{ij}}{\partial x} - m_{i+k,j+l} \frac{\partial^2 u_{ij}}{\partial t^2} + \left(ikM_{i+k-2,j+l} + jlM_{i+k,j+l-2} \right) u_{ij} \right] + X_{kl} = 0,$$

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \left[\frac{\partial}{\partial x} \left(M_{i+k,j+l} \frac{\partial v_{ij}}{\partial x} + iM_{i+k-1,j+l} u_{ij} \right) - kL_{i+k-1,j+l} \frac{\partial u_{ij}}{\partial x} \right.$$

$$-m_{i+k,j+l} \frac{\partial^2 v_{ij}}{\partial t^2} + \left(ikN_{i+k-2,j+l} + jlM_{i+k,j+l-2} \right) v_{ij} \tag{4}$$

$$\left. + \left(jkL_{i+k-1,j+l-1} + ilM_{i+k-1,j+l-1} \right) w_{ij} \right] + Y_{kl} = 0,$$

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left[\frac{\partial}{\partial x}\left(M_{i+k,j+l}\frac{\partial w_{ij}}{\partial x}+jM_{i+k,j+l-1}u_{ij}\right)-lL_{i+k,j+l-1}\frac{\partial u_{ij}}{\partial x}\right.$$

$$-m_{i+k,j+l}\frac{\partial^2 w_{ij}}{\partial t^2}+\left(ilL_{i+k-1,j+l-1}+jkM_{i+k-1,j+l-1}\right)v_{ij}$$

$$\left.+\left(jlN_{i+k,j+l-2}+ikM_{i+k-2,j+l}\right)w_{ij}\right]+Z_{kl}=0, \qquad k=0,1,\dots; \quad l=0,1,\dots,$$

where

$$L_{ij}(x)=\iint_S \frac{E\nu y^i z^j}{(1+\nu)(1-2\nu)}\,ds, \qquad M_{ij}(x)=\iint_S \frac{Ey^i z^j}{2(1+\nu)}\,ds,$$

$$N_{ij}(x)=\iint_S \frac{E(1-\nu)y^i z^j}{(1+\nu)(1-2\nu)}\,ds, \qquad m_{ij}(x)=\iint_S \rho y^i z^j\,ds. \tag{5}$$

(E is the modulus of elasticity, ν is Poisson' s ratio, ρ is the density).

The equations (4) must be supplemented by initial conditions and boundary conditions, determined by the forces acting on the end faces of the rod, or by the constraints imposed on the elastic displacements by one or another fixing of the ends of the rod.

If in formulas (3) and equations (4) we set identically equal to zero the functions u_{ij}, v_{ij}, w_{ij} for which $i+j \geq n$, then the errors obtained in determining the elastic displacements u_x, u_y, u_z by solving the shortened system of differential equations (4) will tend to zero as $n \to \infty$. For $n = 2$, restricting ourselves for simplicity to the case when, by virtue of the symmetry of the rod, the coefficients L_{ij}, M_{ij}, N_{ij} and m_{ij} are identically equal to zero for odd values of the index i and for odd values of the index j, and introducing the notation

$$u=u_{00}, \quad v=v_{00}, \quad w=w_{00}, \quad \varphi=\frac{1}{2}(w_{10}-v_{01}), \quad \psi=u_{01}, \quad \vartheta=-u_{10},$$

$$\xi=\frac{1}{2}(w_{01}+v_{10}), \quad \eta_1=\frac{1}{2}(w_{01}-v_{10}), \quad \eta_2=\frac{1}{2}(w_{10}+v_{01}); \tag{6}$$

$$q_x=X_{00}, \quad q_y=Y_{00}, \quad q_z=Z_{00}, \quad m_x=Z_{10}-Y_{01}, \quad m_y=X_{01},$$
$$m_z=-X_{10}, \quad r=Z_{01}+Y_{10}, \quad s_1=Z_{01}-Y_{10}, \quad s_2=Z_{10}+Y_{01}; \tag{7}$$

$$B = \iint_S \frac{E}{2(1+\nu)}\,ds, \quad B_0 = \iint_S \frac{E(y^2+z^2)}{2(1+\nu)}\,ds, \quad B_x = \iint_S \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\,ds,$$

$$B_y = \iint_S \frac{E(1-\nu)z^2}{(1+\nu)(1-2\nu)}\,ds, \quad B_z = \iint_S \frac{E(1-\nu)y^2}{(1+\nu)(1-2\nu)}\,ds,$$

$$C = \iint_S \frac{E\nu}{(1+\nu)(1-2\nu)}\,ds, \tag{8}$$

$$C_0 = \iint_S \frac{E(z^2-y^2)}{2(1+\nu)}\,ds, \quad m = \iint_S \rho\,ds, \quad I_y = \iint_S \rho z^2\,ds, \quad I_z = \iint_S \rho y^2\,ds,$$

one can represent the shortened system of equations (4) in the form

$$\frac{\partial}{\partial x}\left(B_x\frac{\partial u}{\partial x} + 2C\xi\right) - m\frac{\partial^2 u}{\partial t^2} + q_x = 0,$$

$$\frac{\partial}{\partial x}\left(B_0\frac{\partial \xi}{\partial x} + C_0\frac{\partial \eta_1}{\partial x}\right) - 2C\left(\frac{\partial u}{\partial x} + \xi\right) - 2B\xi - (I_y+I_z)\frac{\partial^2 \xi}{\partial t^2}$$
$$+(I_z-I_y)\frac{\partial^2 \eta_1}{\partial t^2} + r = 0, \tag{9}$$

$$\frac{\partial}{\partial x}\left(B_0\frac{\partial \eta_1}{\partial x} + C_0\frac{\partial \xi}{\partial x}\right) - 4B\eta_1 - (I_y+I_z)\frac{\partial^2 \eta_1}{\partial t^2} + (I_z-I_y)\frac{\partial^2 \xi}{\partial t^2} + s_1 = 0;$$

$$\frac{\partial}{\partial x}\left(B_0\frac{\partial \varphi}{\partial x} - C_0\frac{\partial \eta_2}{\partial x}\right) - (I_y+I_z)\frac{\partial^2 \varphi}{\partial t^2} - (I_z-I_y)\frac{\partial^2 \eta_2}{\partial t^2} + m_x = 0, \tag{10}$$

$$\frac{\partial}{\partial x}\left(B_0\frac{\partial \eta_2}{\partial x} - C_0\frac{\partial \varphi}{\partial x}\right) - 4B\eta_2 - (I_y+I_z)\frac{\partial^2 \eta_2}{\partial t^2} - (I_z-I_y)\frac{\partial^2 \varphi}{\partial t^2} + s_2 = 0;$$

$$\frac{\partial}{\partial x}\left(B_z\frac{\partial \vartheta}{\partial x}\right) + B\left(\frac{\partial v}{\partial x} - \vartheta\right) - I_z\frac{\partial^2 \vartheta}{\partial t^2} + m_z = 0, \tag{11}$$

$$\frac{\partial}{\partial x}\left[B\left(\frac{\partial v}{\partial x} - \vartheta\right)\right] - m\frac{\partial^2 v}{\partial t^2} + q_y = 0;$$

$$\frac{\partial}{\partial x}\left(B_y\frac{\partial \psi}{\partial x}\right) - B\left(\frac{\partial w}{\partial x} + \psi\right) - I_y\frac{\partial^2 \psi}{\partial t^2} + m_y = 0, \tag{12}$$

$$\frac{\partial}{\partial x}\left[B\left(\frac{\partial w}{\partial x}+\psi\right)\right]-m\frac{\partial^2 w}{\partial t^2}+q_z=0.$$

In the case under consideration, formulas (3), according to (6), can be given the form

$$u_x=u-\vartheta y+\psi z,\quad u_y=v+(\xi-\eta_1)y+(\eta_2-\varphi)z,$$
$$u_z=w+(\varphi+\eta_2)y+(\xi+\eta_1)z.$$

The coefficients φ, ψ, ϑ determine the angles of rotation of the cross section of the rod with respect to the axes x, y, z; the coefficients ξ, η_1, η_2 determine the plane deformations of the cross section. The deformation corresponding to the coefficient ξ reduces to a change in the area of the section; under the deformations corresponding to the coefficients η_1 and η_2, the configuration of the transverse–

of the section. The functions $q_x, q_y, q_z, m_x, m_y, m_z$ are distributed forces and moments, the coefficient m is the mass per unit length, and the coefficients I_y, I_z are the mass moments of inertia of the cross-section.

Equations (9) and (10) determine the longitudinal and torsional vibrations of the rod, accompanied by plane deformations of the cross-sections. For $C_0(x)=0$, $I_y(x)=I_z(x)$ (for example, in the case of axial symmetry of the rod), the first of equations (10) becomes the usual equation of torsional vibrations; for $\nu=0$, the first of equations (9) becomes the usual equation of longitudinal vibrations of a rod. Equations (11) and (12) are the ordinary equations of flexural vibrations accompanied by shear; the difference consists only in the formulas for the flexural rigidities B_y and B_z. The formulas (8) for the rigidities B_y and B_z reduce to the usual ones when $\nu=0$.

By extending the shortened system (4) considered by us, we obtain the differential equations of vibrations of an elastic rod that take into account not only plane deformations of its cross-sections, but also their deplanation. Passing to curvilinear coordinates, one may apply the method indicated by us to static and dynamic problems in the theory of curvilinear elastic rods.

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2. V. Z. Vlasov, *Thin-Walled Elastic Rods*, 1959.