P. L. Ulyanov

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Abstract

Full Text

P. L. Ulyanov

ON SERIES IN THE HAAR SYSTEM

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Trigonometric series have been and remain the object of numerous investigations, both from the point of view of their convergence in one sense or another, and from the point of view of the behavior of their coefficients.

In the present note we shall give a number of assertions concerning the behavior of series in the Haar system $\{\chi_m(t)\}$ (see (^{1}), pp. 46–50). These assertions show that series in the Haar system possess certain properties that are fundamentally different from the properties of trigonometric series. Moreover, we shall see that in some questions trigonometric series behave better than series in the Haar system, while in other questions the reverse is true.

§ 1. In this section we shall consider the behavior of the Fourier coefficients of functions expanded in a series in the Haar system.

Let (see (^{1}), p. 46) $\chi_1(t) \equiv \chi_0^{(0)}(t)$ and $\chi_m(t) = \chi_n^{(k)}(t)$ $(t \in [0,1])$ for $m = 2^n + k$, where $1 \le k \le 2^n$ and $n = 0, 1, \ldots$.

Theorem 1. If the function $f(t) \in L^p(0,1)$ for some $p \in [1,\infty]^*$, then

$$|a_m(f)| = \left|\int_0^1 f(t)\chi_m(t)\,dt\right| \le m^{1/p-1/2}\omega_p\left(\frac{1}{m}, f\right) \quad (m > 1), \tag{1}$$

where

$$\omega_p(\delta, f) = \sup_{0\le h\le\delta}\left\{\int_0^{1-h} |f(t+h) - f(t)|^p\,dt\right\}^{1/p} \quad \text{for } 1 \le p < \infty;$$

$$\omega(\delta, f) = \omega_\infty(\delta, f) = \sup_{|x-y|\le\delta} |f(x) - f(y)| \quad \text{for } p = \infty.$$

Estimate (1) is sharp in order.

Corollary 1. If $f(t) \in L^p(0,1)$ for some $p \in [1,\infty]$, then

$$a_m(f) = o\left(m^{1/p-1/2}\right).$$

Next, if we take the class of functions of bounded variation on $[0,1]$, then the following holds for it.

Theorem 2. If $f(t) \in V(0,1)$, then

$$\sum_{m=2^n+1}^{2^{n+1}} |a_m(f)| \leq \frac{3 \overset{1}{\underset{0}{\mathrm{V}}} f}{2\sqrt{2^n}} \quad (n \geq 0); \tag{2}$$

$$|a_m(f)| \leq \frac{3 \overset{1}{\underset{0}{\mathrm{V}}} f}{\sqrt{m}} \quad (m > 1). \tag{3}$$

Estimate (2) is sharp in order. Moreover, estimate (3) cannot be improved either, since there exists a function $f_0(t) \in V(0,1)$ such that

$$a_{m_k}(f_0) > \frac{1}{\sqrt{m_k}} \tag{3'}$$

for some $m_k \uparrow \infty$.

* By the space $L^\infty(0,1)$ we mean the space $C(0,1)$.

§ 2. It is known (see $(^2)$, p. 121) that if the function $f(x) \in V(0, 2\pi)$, then its trigonometric Fourier series converges at each point x_0 to the value

$$\frac{f(x_0+0)+f(x_0-0)}{2}.$$

As for series with respect to the Haar system, here the situation is different, since the following is true.

Theorem 3. There exists a function $f(t) \in V(0,1)$ such that its Fourier series

$$\sum_{m=1}^{\infty} a_m(f)\chi_m(t) \tag{4}$$

diverges on an everywhere dense subset of $[0,1]$.

Moreover, the following is true.*

Theorem 4. If $f(t) \in V(0,1)$, then its Fourier series (4) has the following properties:

a) the series (4) converges to $f(t)$ at all points of continuity of the function $f(t)$;

b) the series (4) converges at all dyadic-rational points;

c) the series (4) diverges at every point of discontinuity of the function $f(t)$, if this point is not dyadic-rational.

Thus, we see that the trigonometric Fourier series of functions of bounded variation behave better than the corresponding series with respect to the Haar system. On the other hand, if we consider continuous functions, then their trigonometric Fourier series may diverge at individual points (du Bois-Reymond), whereas the corresponding series with respect to the Haar system must always be uniformly convergent.

§ 3. Ciesielski and Musielak (^{5}) proved that if $f(t) \in V(0,1)$ and $f(t) \in \operatorname{Lip}\alpha$ with some $\alpha > 0$, then

$$\sum_{m=1}^{\infty} |a_m(f)| = \sum_{m=1}^{\infty} \left| \int_0^1 f(t)\chi_m(t)\,dt \right| < \infty,$$

i.e. an assertion analogous to Zygmund' s theorem for trigonometric series is valid (see (^{3}), p. 138).

The assertion of Ciesielski and Musielak does not reveal any special features of series with respect to the Haar system in this question. In fact, the following is true.

Theorem 5. If $f(t) \in V(0,1)$, then

$$\sum_{m=1}^{\infty} |a_m(f)| \le M + \frac{3}{2-\sqrt{2}}\, V_0^1 f,$$

where

$$M = \sup_{0 \le t \le 1} |f(t)|.$$

Moreover, the following holds.

Theorem $5'$. If $f(t) \in V(0,1)$, then

$$\sum_{m=1}^{\infty} |a_m(f)|^{\alpha} < \infty \quad \text{for every } \alpha > \frac{2}{3}; \tag{5}$$

$$\sum_{m=1}^{\infty} m^{\beta} |a_m(f)| < \infty \quad \text{for every } \beta < \frac{1}{2}. \tag{5'}$$

In addition, there exists a function $f_0(t) \in V(0,1)$ such that

$$\sum_{m=1}^{\infty} |a_m(f_0)|^{2/3} = \sum_{m=1}^{\infty} m^{1/2} |a_m(f_0)| = \infty.$$

Thus, although the Fourier coefficients with respect to the Haar system of functions of bounded variation have the unimprovable order $m^{-1/2}$ (see (3) and (3′)), nevertheless, in the main they have order $m^{-3/2}$ (see (5) and (5′)).

Let us note one more result concerning absolute convergence of series with respect to the Haar system, namely, the following is valid (see (^{4}))

* **Note added in proof.** After the article had been submitted for publication, it became known to the author (see (1), p. 257) that Theorem 4 had in fact been established by Faber.

Theorem 6. If a_m are such that

$$\sum_{m=1}^{\infty} \frac{|a_m|}{\sqrt{m}} < \infty,$$

then the series

$$\sum_{m=1}^{\infty} a_m \chi_m(t) \tag{6}$$

converges absolutely almost everywhere on $[0, 1]$.

From this theorem the following follow immediately:

Corollary 2. If, for some $\varepsilon > 0$,

$$\sum_{m=1}^{\infty} a_m^2 \lg^{1+\varepsilon} m < \infty,$$

then the series (6) converges absolutely for almost all $t \in [0, 1]$.

Corollary 3. If a function $f(t)$ is such that, for some $\varepsilon > 0$,

$$\omega_1(\delta, f) = O\left(\lg^{-1-\varepsilon} \frac{1}{\delta}\right) \qquad (\delta \to +0),$$

then the Fourier series of the function $f(t)$ converges absolutely almost everywhere on $[0, 1]$.

§ 4. In this section we shall give assertions concerning series with monotone coefficients.

Theorem 7. If $a_m \downarrow 0$, then in order that the series (6) be absolutely and uniformly convergent on $[0,1]$, it is necessary and sufficient that one of the following conditions hold:

a)

$$\sum_{m=1}^{\infty} \frac{a_m}{\sqrt{m}} < \infty;$$

b) the series

$$\sum_{m=1}^{\infty} |a_m \chi_m(t)|$$

converges at at least one point $t_0 \in [0,1]$;

c) the series (6) is the Fourier series of a bounded function.

Moreover, if condition a), or b), or c) is fulfilled, then the series (6) is the Fourier series of some bounded function having no more than a countable number of discontinuity points.

Theorem 8. If $a_m \downarrow 0$, then in order that the series (6) be unconditionally convergent almost everywhere on $[0,1]$, it is necessary and sufficient that inequality a) of Theorem 7 hold.

Theorem 9. If $a_m \downarrow 0$, $p > 1$, then from the fact that the series (6) is the Fourier series of a function $f(t) \in L^p(0,1)$, it follows that $f(t) \in L^\alpha(0,1)$ for all $\alpha \geq 1$, and the series (6) is a Fourier series of class L^α for all $\alpha \in [1,\infty)$.

§ 5. Schauder $^{(6)}$ proved that if $f(t) \in L^p(0,1)$ for some $p \in [1,\infty)$, then the Fourier series of f with respect to the Haar system is convergent (to f) in the norm of the space $L^p(0,1)$. In fact, a stronger assertion is true. Namely, we have

Theorem 10. If $f(t) \in L^p(0,1)$ for some $p \in [1,\infty)$, then for $m \geq 1$

$$\left\| f(t) - \sum_{\nu=1}^{m} a_\nu(f)\chi_\nu(t) \right\|_p \leq 24\, \omega_p \left(\frac{1}{m}, f \right). \tag{7}$$

We note that, for the space $C(0,1)$, an estimate of the form (7) was proved by Hadem (see $^{(1)}$, pp. 294-295).

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References Cited

[1] G. Alexits, *Convergence Problems of Orthogonal Series*, Budapest, 1961.
[2] N. K. Bari, *Trigonometric Series*, Moscow, 1961.
[3] A. Zygmund, *Trigonometric Series*, Moscow-Leningrad, 1939.
[4] P. I. Ulyanov, UMN, 16, no. 3, 61 (1961).
[5] L. Ciesielski, J. Musielak, *Colloq. Math.*, 7, no. 1, 61 (1959).
[6] J. Schauder, *Math. Zs.*, 28, 317 (1928).

Note: Figure translations are in progress. See original paper for figures.

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