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Abstract

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ON A CERTAIN PROPERTY OF THE LOCATION OF SINGULARITIES ON THE BOUNDARY OF A POLY-CYLINDER AND ITS APPLICATION TO ENTIRE FUNCTIONS OF SEVERAL VARIABLES

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For the case of entire functions of finite degree* of two variables, V. K. Ivanov ([3]) obtained the following generalization of G. Pólya' s well-known theorem on the relation between the conjugate and indicator diagrams.

Theorem A. Let

$$f(z_1, z_2) = \sum_{k_1=0,\ k_2=0}^{\infty} \frac{a_{k_1,k_2}}{k_1!\, k_2!} z_1^{k_1} z_2^{k_2}$$

be an entire function of finite degree,

$$F(z_1, z_2) = \sum_{k_1=0,\ k_2=0}^{\infty} \frac{a_{k_1,k_2}}{z_1^{k_1+1} z_2^{k_2+1}}$$

the function associated with the function $f(z_1, z_2)$. Suppose, further, that by $T(\varphi_1, \varphi_2)$ is denoted the set of those points (v_1, v_2) of the real plane such that, for some constant $A(v_1, v_2)$, for all nonnegative r_1, r_2 the inequality

$$\left| f\left(r_1 e^{i\varphi_1}, r_2 e^{i\varphi_2}\right) \right| \leq A(v_1, v_2) e^{v_1 r_1 + v_2 r_2},$$

holds, and let $C(\varphi_1, \varphi_2)$ be the set of such points (v_1, v_2) that $F(z_1, z_2)$ is analytic in the domain

$$\mathrm{Re}\{z_1 \exp(-i\varphi_1)\} > v_1, \qquad \mathrm{Re}\{z_2 \exp(-i\varphi_2)\} > v_2.$$

Then

$$\overline{C}(-\varphi_1, -\varphi_2) = \overline{T}(\varphi_1, \varphi_2).$$

The central point in V. K. Ivanov' s proof of this theorem is

Theorem B (Lemma 3 from ([3])). Let the function $g(z_1, z_2)$ be regular inside the bicylinder $\{|z_1| < r_1,\ |z_2| < r_2\}$. Suppose, in addition, that $g(z_1, z_2)$ is regular at all points of the lateral surface of the bicylinder whose coordinates satisfy the conditions $|z_1| = r_1$, $|z_2| \le r_2$, but $z_2 \ne r_2$. Then, if the point (z_1^0, r_2), where $|z_1^0| < r_1$, is a singular point of the function $g(z_1, z_2)$, then all points (z_1, r_2) with $|z_1| \le r_1$ are singular for the function $g(z_1, z_2)$.

It seems to us that this fact, concerning the location of the singularities of an analytic function of two variables, is of importance independently of the generalization of Pólya' s theorem.

M. Sh. Stavskii ([4]) transferred Theorem A to another case. N. I. Kobeleva ([5]) obtained a certain analogue of this theorem for the case of entire functions of two variables of finite order (ρ_1, ρ_2) in the sense of M. M. Dzhrbashyan ([6]), i.e., for the case of entire functions of finite order of growth whose conjugate orders and types** are curves that are the boundaries of quadrants. Let us note that also in ([4,5]), in the proofs of the assertions playing the same role as the above Theorem B (Lemma 4 in ([4]), Lemma 3* in ([5])), there are errors which we have been unable to eliminate.

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* For entire functions of finite degree of several variables, see ([1]), ([2]), § 26.
** On conjugate orders and conjugate types, see ([2]), § 26.

In our note, using a method different from that of V. K. Ivanov, we shall obtain, for the case of n variables, an assertion which for $n = 2$ is a certain generalization of Theorem B. This will then make it possible to obtain, for arbitrary entire functions of finite order in n variables, a theorem which, on the one hand, is a certain analogue of Theorem A, and on the other hand is a generalization, to the case of an arbitrary number of variables, of the results of A. A. Avetisyan ([7]) and I. F. Lokhina ([8]), concerning entire functions of one variable. In particular, this yields the results of M. Sh. Stavskii and N. I. Kobeleva.

Theorem 1. *Let the function $g(z_1, \dots, z_n)$ be analytic in the polycylinder $D = \{z_i \in D_i;\ i = 1, 2, \dots, n\}$, where D_i is a domain in the plane of the variable z_i, $i = 1, 2, \dots, n$. Suppose, further, that the point $z^0 = (z_1^0, \dots, z_n^0)$, a boundary point of the polycylinder D, is a singular point for the function $g(z_1, \dots, z_n)$, and denote by P the set of those values of the index i for which $z_i^0 \in D_i$.*

Then all points of the manifold

$$D_0 = \{z_i = z_i^0,\ z_i \in D_i\}_{i \notin P}^{i \in P}$$

are singular points of the function $g(z_1, \dots, z_n)$.

Proof. Denote by $d(z, \zeta)$ the distance between the points $z = (z_1, \dots, z_n)$ and $\zeta = (\zeta_1, \dots, \zeta_n)$, defining it as follows:

$$d(z, \zeta) = \max_i |z_i - \zeta_i|.$$

Denote by $\Delta_g(z)$ the distance from the point $z \in D$ to the nearest singular point of the function $g(z_1, \dots, z_n)$. It is known (see, for example, ([9])) that $-\ln \Delta_g(z)$ is a plurisubharmonic function of the variables $z_1, \dots, z_n$, and, by virtue of a known property of such functions, it will also be plurisubharmonic on every analytic manifold lying in D. Let us now choose, in each domain D_i for $i \in P$, an open set $D_{i,r}$ defined by the condition

$$\min_{z_i^* \in F(D_i)} |z_i^* - z_i| > r,$$

where $F(D_i)$ denotes the boundary of the domain D_i, and $r > 0$ is taken so small that $z_i^0 \in D_{i,r}$. Further, for $i \notin P$, choose in each domain D_i a point $z_i^{(r)}$ such that $|z_i^{(r)} - z_i^0| = r$. Denote by D_r the set of those points $z = (z_1, \dots, z_n)$ such that $z_i = z_i^{(r)}$ for $i \notin P$, and $z_i \in D_{i,r}$ for $i \in P$. In view of what was said above about the plurisubharmonicity of the function $-\ln \Delta_g(z)$, this function is, on D_r, a plurisubharmonic function of the variables z_i, where $i \in P$. Since the maximum principle holds for plurisubharmonic functions, the minimum principle will hold for the function $\Delta_g(z)$ on D_r, i.e., $\min_{D_r} \Delta_g(z)$ can be attained at an interior point of D_r only when

$$\Delta_g(z) \equiv \min_{D_r} \Delta_g(z).$$

From the definition of the manifold D_r, and from the fact that the function $g(z_1, \dots, z_n)$ is analytic in D, it follows that on D_r, $\Delta_g(z) \geq r$. At the same time, at least at one of the interior points of D_r, $\Delta_g(z) = r$. Indeed, the point z^*, whose coordinates are defined as follows: $z_i^* = z_i^{(r)}$ for $i \notin P$, $z_i^* = z_i^0$ for $i \in P$, obviously belongs to D_r, and $d(z^0, z^*) = r$. Applying the minimum principle, we further conclude that $\Delta_g(z) \equiv r$ on D_r.

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* For $n = 2$ and $D_i = (|z_i| < R_i)$, Theorem 1 is, in essence, equivalent to Theorem B. Let us note that even in the case when all D_i are simply connected domains, Theorem 1 cannot be obtained from Theorem B by means of a conformal mapping of the domains D_i onto circles.

If we now let r tend to zero, then we obtain that in an arbitrarily small neighborhood of each point of the manifold $\overline{D}_0$ there are singular points of the function $g(z_1, \dots, z_n)$. Consequently, every point of this manifold is singular. The theorem is proved.

Let now

$$f(z) = \sum_{k_1=0,\dots,k_n=0}^{\infty} \frac{a_{k_1,\dots,k_n}}{\Gamma(1 + k_1\rho_1^{-1}) \cdots \Gamma(1 + k_n\rho_n^{-1})} z_1^{k_1} \cdots z_n^{k_n} \tag{1}$$

be an entire function of finite order of growth, and let $(\rho_1, \dots, \rho_n)$ be a certain system of its conjugate orders, with $\rho_i \geq \frac{1}{2}$, $i = 1, \dots, n$. Introduce for consideration the function associated with it,

$$F(z) = \sum_{k_1=0,\dots,k_n=0}^{\infty} \frac{a_{k_1,\dots,k_n}}{z_1^{k_1+1} \dots z_n^{k_n+1}}.$$

Just as is done for the case of entire functions of finite degree (see ([2])), it is easy to show that the surface formed by the points $(\sigma_1^{1/\rho_1}, \dots, \sigma_n^{1/\rho_n})$, where $(\sigma_1, \dots, \sigma_n)$ is a point on the surface of conjugate types of order $(\rho_1, \dots, \rho_n)$ of the function $f(z)$, coincides with the surface of conjugate radii of convergence of the function $F(z)$. For a more precise characterization of the relation between the growth of the function $f(z)$ and the disposition of the singularities of the function $F(z)$, consider the set $T(\varphi)$ of points $(\nu_1, \dots, \nu_n)$ with $\nu_i > 0$, $i = 1, \dots, n$, for which, with some constant $A(\nu)$ and all nonnegative $r_1, r_2, \dots, r_n$, the inequality

$$\left|f(r_1 e^{i\varphi_1}, \dots, r_n e^{i\varphi_n})\right| < A(\nu) \exp(\nu_1 r_1^{\rho_1} + \dots + \nu_n r_n^{\rho_n}),$$

holds; and also the set $C(\varphi)$ of points $(\nu_1, \dots, \nu_n)$ with $\nu_i > 0$, for which the function $F(z)$ is analytic in the domain

$$\left\{ \operatorname{Re}(z_j e^{-i\varphi_j})^{\rho_j} > \nu_j,\ |\arg z_j - \varphi_j| < \frac{\pi}{2\rho_j};\ j = 1, \dots, n \right\}.$$

We note that in the case where $\rho_j = 1$, $j = 1, \dots, n$, in the definition of the sets $T(\varphi)$ and $C(\varphi)$ the requirement that $\nu_1, \dots, \nu_n$ be positive is removed.

Theorem 2. *For every entire function (1), the sets $\overline{T}(\varphi)$ and $\overline{C}(-\varphi)$ coincide.*

We briefly outline the proof of the theorem. First of all, note that the following integral representations hold, proved in exactly the same way as in the case of one variable (see, for example, ([10])):

$$f(z) = \frac{1}{(2\pi i)^n} \int_{l_1} \cdots \int_{l_n} F(\zeta_1, \dots, \zeta_n) \prod_{j=1}^{n} E_{\rho_j}(z_j \zeta_j)\, d\zeta_j, \tag{2}$$

where

$$E_\rho(\zeta_j) = \sum_{k=0}^{\infty} \frac{\zeta_j^k}{\Gamma(1 + k\rho^{-1})},$$

and the closed contours l_j are such that the function $F(z)$ is analytic when z_j lies on the contour l_j or outside it,

$$F(z)=\prod_{j=1}^{n}\left\{\frac{\rho_j}{z_j}(e^{-i\varphi_j}z_j)^{\rho_j}\right\}\int_0^{\infty}\cdots\int_0^{\infty}f(t_1e^{-i\varphi_1},\dots,t_ne^{-i\varphi_n})\times$$

$$\times\prod_{j=1}^{n}t_j^{\rho_j-1}\exp\{-t_j^{\rho_j}(e^{-i\varphi_j}z_j)^{\rho_j}\}\,dt_j. \tag{3}$$

We shall now denote by $l_j(R,\varepsilon)$ the closed contour formed by an arc of the circle $|z_j|=R$ and an arc of the curve

$$\operatorname{Re}(z_je^{-i\varphi_j})^{\rho_j}=v_j+\varepsilon,\qquad |\arg z_j-\varphi_j|<\frac{\pi}{2\rho_j}.$$

Lemma. If $(v_1,\dots,v_n)\in C(\varphi)$, then for every $\varepsilon>0$ there is an $R(\varepsilon)$ such that the function $F(z)$ will be analytic under the condition that z_j lies on the contour $l_j(R(\varepsilon),\varepsilon)$ or outside it.

Proof. Denote

$$D_\varepsilon=\left\{\operatorname{Re}(w_je^{i\varphi_j})^{-\rho_j}>v_j+\varepsilon,\ |\arg w_j+\varphi_j|<\frac{\pi}{2\rho_j},\ j=1,\dots,n\right\}.$$

Suppose that in D_ε there is a point $(w_1^0,\dots,w_n^0)$ singular for the function

$$g(w)=F\left(\frac{1}{w_1},\dots,\frac{1}{w_n}\right).$$

Denote by Q the set of such values of the index j that $w_j^0=0$ for $j\in Q$. Since $(v_1,\dots,v_n)\in C(\varphi)$, the set Q is nonempty. Hence, by Theorem 1 we conclude that all points of the manifold

$$N=\left\{w_j=w_j^0,\ j\in Q;\quad w_j\in\left(\operatorname{Re}(w_je^{i\varphi_j})^{-\rho_j}>v_j\right)_{j\notin Q}\right\}$$

are singular. But this contradicts the holomorphy of the function $g(w)$ at the point $(0,\dots,0)\in\overline{N}$. Consequently, in $\overline{D}_\varepsilon$ there are no singular points of the function $g(w)$. The assertion of the lemma follows from this by passing to the variables $z_j=w_j^{-1}$.

From this lemma it follows that in (2), as the contours l_j, one may take the contours $l_j(R(\varepsilon),\varepsilon)$. Further, estimating the integral standing on the right-hand side of (2), similarly to how this is done in the case of one variable (7), we conclude that

$$C(-\varphi)\subset\overline{T}(\varphi).$$

At the same time, from (3), by a direct estimate of the right-hand side we obtain the inclusion

$$T(\varphi) \subset \overline{C}(-\varphi),$$

therefore,

$$\overline{T}(\varphi) = \overline{C}(-\varphi).$$

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Note: Figure translations are in progress. See original paper for figures.