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Abstract

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ON LINEAR ELLIPTIC SYSTEMS OF FIRST ORDER

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This note investigates two questions.

First the structure of elliptic systems of first order with real coefficients in the k-dimensional Euclidean space R_k is clarified. It turns out that to each $k \geq 2$ there corresponds a certain number $n(k)$ such that: 1) in R_k there exist elliptic systems of $n(k)$ equations of first order; 2) whatever elliptic system of first order in R_k is taken, the number of equations in it is a multiple of $n(k)$. Among systems of $n(k)$ equations of first order there are, in particular, such systems for which the solutions of the homogeneous system are harmonic vectors. Such systems will be called systems of the simplest structure in R_k.

In the second part of the note a system of the simplest structure in R_4 is considered. We shall show that this system has not a single correct integro-differential boundary-value problem in any bounded domain.

1. Let $A\left(\frac{1}{i}\frac{\partial}{\partial x}\right)$ be a matrix of size $n \times n$, the entries of which are first-order differential operators in the space R_k with constant coefficients. In Fourier images this differential matrix corresponds to the matrix $A(\alpha)$, the entries of which are real linear forms with respect to $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$. The determinant $\det A(\alpha)$ is a form of order n with respect to α. The matrix $A\left(\frac{1}{i}\frac{\partial}{\partial x}\right)$ is assumed to be elliptic in the sense of I. G. Petrovsky. Then the form $\det A(\alpha)$ must be positive definite:

$$\det A(\alpha) > 0 \quad (\alpha \neq 0). \tag{1}$$

Condition (1) imposes very strict restrictions which must be satisfied by the dimension of the space k and the number of equations in the system n. Namely, denote by $\rho(n)$ the number defined as follows: if n is written in the form $n = (2a+1) \cdot 2^{4b+c}$ $(0 \leq c \leq 3)$, then $\rho(n) = 8b + 2^c$.

Theorem. *Elliptic systems of n equations of first order with real coefficients in the space R_k exist if and only if*

$$k \leq \rho(n).$$

From this theorem there follows immediately the assertion formulated at the beginning of the article. In this case

$$n(k) = \min_{\rho(n) \geq k} n.$$

The proof of the theorem uses some results from the theory of frame fields on spheres. It turns out that every matrix $A(\alpha)$ satisfying condition (1) gives rise on the $(n-1)$-dimensional sphere S_{n-1} to a system of $(k-1)$ continuous vector fields, linearly independent at every point of the sphere, i.e. to a so-called frame $(k-1)$-field. In Adams' paper ([1]) it is shown that such fields exist only if $k \leq \rho(n)$. On the other hand, we use the following theorem ([2–4]): the product

$$(x_1^2 + x_2^2 + \dots + x_n^2)(\alpha_1^2 + \alpha_2^2 + \dots + \alpha_k^2)$$

can be represented in the form

$$z_1^2 + z_2^2 + \dots + z_n^2, \tag{2}$$

where z_i are bilinear forms in x and α, if and only if $k \leq \rho(n)$. Suppose that this inequality is also satisfied and, in the representation (2),

$$z_i(x, \alpha) = \sum_{j=1}^{n} L_{ij}(\alpha) x_j.$$

As was established in ([4]), there always exist such representations in which the coefficients of the linear forms $L_{ij}(\alpha)$ are real.

It is easy to see that the matrix

$$A(\alpha) = (L_{ij}(\alpha)) \qquad (i, j = 1, 2, \dots, n)$$

is orthogonal. Consequently, the corresponding system of partial differential equations is elliptic, and the solutions of the homogeneous system are harmonic vectors. If $n = n(k)$, then in this way we obtain a system of the simplest structure.

Let us note that the connection of the Hurwitz–Radon–Eckmann theorem both with the theory of frames on spheres and with the problem of the existence of nondegenerate matrices of linear forms was pointed out and used already in the works ([5],[6]).

2. The study of boundary-value problems for elliptic systems of first order should naturally be carried out first of all for systems of the simplest structure. A boundary-value problem is usually defined by prescribing the

values of certain linear combinations of the unknown functions and their derivatives on the boundary of the domain. Recently, boundary conditions of a more general type have also been considered ([7],[8])—the so-called integro-differential (i.-d.) conditions, in which the unknown functions and their derivatives may occur under the sign of singular integrals extended over the boundary of the domain. Apparently, the class of i.-d. conditions is the most natural one for elliptic equations and systems.

Consider in the space R_4 the operator

$$A\left(\frac{1}{i}\frac{\partial}{\partial x}\right) = \frac{1}{i}\begin{pmatrix} \frac{\partial}{\partial x_1} & -\frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_3} & -\frac{\partial}{\partial x_4} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & -\frac{\partial}{\partial x_4} & \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_4} & \frac{\partial}{\partial x_1} & -\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_4} & -\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} \end{pmatrix}. \quad (3)$$

In Fourier transforms it corresponds to the matrix

$$A(\alpha) = \begin{pmatrix} \alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ \alpha_2 & \alpha_1 & -\alpha_4 & \alpha_3 \\ \alpha_3 & \alpha_4 & \alpha_1 & -\alpha_2 \\ \alpha_4 & -\alpha_3 & \alpha_2 & \alpha_1 \end{pmatrix}.$$

This matrix is orthogonal; $\det A(\alpha) = |\alpha|^4$, where

$$|\alpha|^2 = \alpha_1^2 + |\alpha'|^2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2.$$

Since $n(4) = 4$, the operator (3) is of the simplest structure in R_4.

It turns out that for the operator (3) there does not exist a single correct i.-d. boundary-value problem in any bounded domain. Moreover, no i.-d. problem for this system is a Φ-problem*.

* That is, the operator corresponding to the problem either is not normally solvable, or the range of this operator or of its adjoint has infinite defect; see ([9]).

In connection with this it should be noted that the known system of A. V. Bitsadze [(10)] (of second order), for which the first boundary-value problem is not correct, has differential boundary-value problems that are Φ-problems. Let us also note that, in the sense of extension theory, every linear elliptic system in the sense of I. G. Petrovsky with constant coefficients has correct boundary-value problems (see, for example, the criterion established in [(11)]).

We shall show that the operator (3) has no correct boundary integro-differential problems in any half-space of the space R_4; for the proof it is enough to verify that the well-known condition of Ya. B. Lopatinskii [(12)] is not satisfied. Hence, as established in [(8)], it follows that this operator has no boundary integro-differential Φ-problems in any finite domain.

We shall restrict ourselves to verifying the Lopatinskii condition for the half-space $x_1 \geq 0$. It is not difficult to show that the general case reduces to this one. Since the operator (3) is of first order, it may be assumed that the boundary conditions contain no differentiation with respect to x_1. Taking this into account, we write the principal part of the matrix of the boundary conditions in Fourier images in the form

$$B(\alpha') = \begin{pmatrix} M_1(\alpha') & M_2(\alpha') & M_3(\alpha') & M_4(\alpha') \\ N_1(\alpha') & N_2(\alpha') & N_3(\alpha') & N_4(\alpha') \end{pmatrix}.$$

Here $M_i(\alpha')$, $N_i(\alpha')$ are real continuous homogeneous functions of their arguments; if the boundary conditions are purely differential, then these functions are polynomials. The Lopatinskii condition consists in the requirement that, for every α' on the sphere $|\alpha'| = 1$, the rank of the matrix

$$L(\alpha') = B(\alpha') \int_{C_+} A^{-1}(\alpha_1, \alpha')\, d\alpha_1$$

must be equal to two (C_+ is a contour situated in the upper α_1-half-plane and enclosing all roots of the characteristic equation lying in it). Computing the matrix $L(\alpha')$, one can verify that all its minors of order two vanish if either the system of equations

$$\frac{\alpha_2}{\begin{vmatrix} M_1 & M_2 \\ N_1 & N_2 \end{vmatrix} + \begin{vmatrix} M_3 & M_4 \\ N_3 & N_4 \end{vmatrix}} = \frac{\alpha_3}{\begin{vmatrix} M_1 & M_3 \\ N_1 & N_3 \end{vmatrix} + \begin{vmatrix} M_4 & M_2 \\ N_4 & N_2 \end{vmatrix}} = \frac{\alpha_4}{\begin{vmatrix} M_1 & M_4 \\ N_1 & N_4 \end{vmatrix} + \begin{vmatrix} M_2 & M_3 \\ N_2 & N_3 \end{vmatrix}}, \tag{4}$$

is satisfied, or if all denominators in (4) are simultaneously equal to zero.

Consider on the two-dimensional sphere $|\alpha'| = 1$ the vector field whose components are these denominators. Since the field is continuous, by the hedgehog theorem (see, for example, [(13)], p. 584), there exists on the sphere a point at which the vector of the field is either equal to zero or is parallel to the normal; the latter means that the system (4) is satisfied. In any of these cases, at the point found the Lopatinskii condition is violated, which was required to be proved.

In the example analyzed, both the unknown functions and the coefficients of the system and of the boundary conditions (in Fourier images) are real. It is easy to construct a system in which the unknown functions are complex and which has no complex integro-differential Φ-problems. Thus, this property is possessed by the system

$$\frac{\partial v_1}{\partial x_1} + i\frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_3} - i\frac{\partial v_2}{\partial x_4} = f_1(x);$$

$$\frac{\partial v_1}{\partial x_3} - i\frac{\partial v_1}{\partial x_4} + \frac{\partial v_2}{\partial x_1} - i\frac{\partial v_2}{\partial x_4} = f_2(x).$$

Indeed, putting $v_1 = u_1 + iu_2$, $v_2 = u_3 - iu_4$ and separating real and imaginary parts, we arrive at the operator (3).

3. A. A. Desin' s paper (14) considers an important class of elliptic systems of first order in spaces of arbitrary dimension, in a certain sense generalizing the Cauchy—Riemann system in the plane. The solutions of these systems are harmonic vectors. The number of equations in A. A. Desin's system for the space R_k is 2^{k-1}, and for $k > 3$ exceeds $n(k)$. Consequently, for $k > 3$ these systems are not systems of the simplest structure. In this connection it is interesting to note that Desin' s systems always have (see (14)) well-posed boundary-value problems, in which the boundary conditions consist in prescribing the values of certain linear combinations of the unknown functions on the boundary of the domain.

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Note: Figure translations are in progress. See original paper for figures.