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Abstract

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A SOLVABILITY CRITERION FOR THE DIRICHLET PROBLEM FOR ELLIPTIC EQUATIONS

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I. For the quasilinear elliptic equation

$$L(u) \equiv \sum_{ij} a_{ij}(x,u)u_{x_i x_j} + \sum_i a_i(x,u,u_{x_k})u_{x_i} + a(x,u) = 0 \qquad (1)$$

in a bounded domain G of n-dimensional space with boundary S, the Dirichlet problem is posed

$$u\big|_S = \varphi(x). \qquad (2)$$

A criterion is given for the solvability of this problem which is not connected with restrictions on the function $a(x,u)$ (see the theorem). Analogous criteria for particular cases occur in ([4,9]).

On the basis of the criterion obtained, an analysis is carried out of the Dirichlet problem for a simple but practically important equation.

We shall precede the formulation of the main theorem by a lemma of the type of comparison theorems concerning parabolic equations. Below we shall use the notation $C^{k+\alpha}$ and $|\cdot|_{k+\alpha}$ for function spaces and the corresponding norms, as well as $A^{k+\alpha}$ for the characteristic of boundaries of domains (see ([2])).

Lemma. Let $L_1(u) = u_t - F(x,t,u,u_{x_k},u_{x_i x_j})$ be a parabolic operator in the domain $Q_T = \{G \times [0,T]\}$, $T < \infty$, and let F have continuous partial derivatives with respect to $u, u_{x_k}, u_{x_i x_j}$ $(i,j,k = 1,\dots,n)$. Suppose there exist two functions $u_1(x,t), u_2(x,t)$ from $C^2(Q_T)$ such that: 1) $L_1(u_1) \le L_1(u_2)$ in Q_T; 2) $u_1|_\Gamma \le u_2|_\Gamma$, where $\Gamma = \overline{G} \cup \{S \times [0,T]\}$.

Then $u_1(x,t) \le u_2(x,t)$ everywhere in the domain Q_T.

This assertion follows from the maximum principle for the equation satisfied by the function $w = u_2 - u_1$ (cf. ([4])).

For the elliptic equation $L_2(u) \equiv F(x, u, u_{x_k}, u_{x_i x_j}) = 0$ with boundary condition $u|_S = \varphi(x)$, we obtain:

Corollary. Suppose there exists a function $v(x,t) \in C^2(Q_T)$ for every $T > 0$, and such that: 1) for all $t \geq 0$, $v|_S \leq \varphi(x)$ $(\geq \varphi(x))$, $v_t - L_2(v) \leq 0$ (≥ 0) in Q_T; 2) along some sequence $\{x_k, t_k\}$, $v(x_k, t_k) \to +\infty$ $(\to -\infty)$.

Then the problem $L_2(u) = 0,\ u|_S = \varphi(x)$ has no solution $u(x) \in C^2(\overline{G})$ satisfying the inequality $u(x) \geq v(x, 0)$ $(\leq v(x,0))$.

Theorem. Suppose that for problem (1)–(2) the following conditions are fulfilled:

A. There exist two functions $v_1(x), v_2(x)$ from $C^2(\overline{G})$, and at least one of them, for example $v_1(x) \in C^{2+\nu}(G)$, $\nu > 0$, such that: 1) $L(v_2) \leq 0 \leq L(v_1)$ in G; $L(v_1) = f_1(x) \in C^{1+\nu}(G)$; 2) $v_1(x) \leq v_2(x)$ in G; 3) $v_1|_S \leq \varphi(x) \leq v_2|_S$.

B. In the domain $G_1 : \{x \in G;\ v_1 \leq u \leq v_2\}$, the ellipticity condition is uniformly satisfied

$$\sum_{ij} a_{ij}(x,u)\xi_i\xi_j \geq a \sum_i \xi_i^2, \qquad a > 0.$$

B. $G \in A^{2+\nu}$; $\varphi(x) \in C^{2+\nu}(S)$; $a_{ij}(x,u) \in C^{2+\nu}(G_1)$; $a(x,u) \in C^{1+\nu}(G_1)$ and, in every compact part of the domain $G_2 : \{(x,u) \in G_1,\ -\infty < p_k < +\infty,\ k = 1, 2, \dots, n\}$, the functions $a_i(x, u, p_1, \dots, p_n) \in C^{1+\nu}$.

C. For $(x,u) \in G_1$ there exists $A > 0$ such that

$$|a_i| + \sum_k \left|\frac{\partial a_i}{\partial x_k}\right| + \left|\frac{\partial a_i}{\partial u}\right| + \left(\sum_k \left|\frac{\partial a_i}{\partial p_k}\right|\right)\left(1 + \sum_k |p_k|\right) \leq A\left(1 + \sum_k |p_k|\right).$$

Under these conditions there exists at least one solution $v(x) \in C^{2+\alpha}(G)$, $\alpha < \nu$, of problem (1), (2), satisfying the inequality

$$v_1(x) \leqslant v(x) \leqslant v_2(x). \tag{3}$$

We note that if the functions a_i do not depend on the derivatives, then the smoothness assumptions on the coefficients can be weakened (for example, it suffices to take a_{ij}, a_i, a from $C^1(G_1)$), and condition C is absent.

The proof of the theorem is based on an a priori estimate $|u|_{2+\gamma}$ in the domain $Q : \{x \in G;\ 0 \leqslant t < \infty\}$, which is obtained for the solution of the boundary-value problem (4), (5)

$$u_t = L(u) - f_1(x)\omega(t); \tag{4}$$

$$u\big|_{t=0} = v_1(x); \qquad u\big|_S = \omega(t)v_1(x)\big|_S + (1-\omega(t))\varphi(x), \tag{5}$$

where $\omega(t)$ is a sufficiently smooth function satisfying the conditions: $0 \leqslant \omega(t) \leqslant 1$, $\omega(0)=1$, $\omega'(0)=\omega'(1)=0$, $\omega'(t)\leqslant 0$, $\omega(t)\equiv 0$ for $t \geqslant 1$. It is introduced in order to match the initial and boundary conditions for $t=0,\ x\in S$.

To obtain the required estimate, consider the sequence of functions $u_k(x,t) = u(x,k+t)$, $0\leqslant t\leqslant 1$, which for $k\geqslant 1$ in the domain $Q_1 : \{x\in G;\ 0<t<1\}$ satisfy the equation

$$(u_k)_t = L(u_k) \tag{6}$$

and the conditions

$$u_k\big|_{t=0} = u_{k-1}(x,1), \qquad u_k\big|_S = \varphi(x). \tag{7}$$

By the lemma, for $u(x,t)$ we have the estimate $v_1(x)\leqslant u(x,t)\leqslant v_2(x)$; consequently, we have an estimate of $|u_k|_0$ in $\overline{Q}_1$, independent of k.

In the domain $\overline{Q}_1$ one can estimate successively, independently of k, the norms $|u_k|_\alpha$, $|u_k|_1$, $|u_k|_{1+\delta}$, $|u_k|_{2+\gamma}$, $\alpha>0$, $\delta>\nu>\gamma$. First, using the results and methods of works ([1−3,5]), we estimate the corresponding norm in the domain $Q_2 : \{x\in G;\ 1/2 < t \leqslant 1\}$, independently of the initial function $u_{k-1}(x,1)$. This gives an estimate, independent of k, for $u_k(\underline{x},1)$. Then, by virtue of the results of works ([1,3,5]), this norm is estimated in $\overline{Q}_1$.

Having such an estimate, one can establish the theorem on the existence of a solution $u(x,t)\in C^{2+\gamma}(Q)$ of problem (4), (5), either by using the Leray–Schauder theorem, as was done in ([2]), or by using Sobolev' s local existence theorem ([6]). The estimate obtained makes it possible to continue the solution indefinitely. Further, from the assumed smoothness of the coefficients there follows the possibility of differentiating with respect to t the derivatives $u_t, u_{x_k}, u_{x_i x_j}$ $(i,j,k=1,2,\dots,n)$. From the equation for u_t, which is obtained by differentiating (4) with respect to t, taking into account the resulting boundary conditions for u_t and the maximum principle, it follows that $u_t\geqslant 0$, i.e. the sequence $u_k(x,t)$ is monotonically increasing in k, and the limiting function v does not depend on t. The estimate obtained for u_k ensures that v belongs to the class $C^{2+\alpha}(G)$ for $\alpha<\gamma$ and makes it possible to pass to the limit along some subsequence in equation (6). Thus we obtain the required solution of problem (1), (2) with condition (3).

II. Usually, in existence theorems, instead of condition A it is required that, for all u, the inequality $a_u \leqslant \beta < 0$ hold. It is easy to see that in this case condition A is satisfied: it suffices to take as the

v_1 and v_2 are constants. However, in a number of problems this requirement is too restrictive. As an example, let us consider the problem

$$L(u) \equiv \Delta u + \lambda F(u) = 0; \quad u|_S = 0. \tag{8}$$

Let the domain G be the unit ball of n-dimensional space, or any subdomain of it. To this type belongs the problem of thermal self-ignition in a closed vessel with an exponentially growing function $F(u)$ ([7], [8]).

Usually one considers the question of the critical dimensions of the vessel, i.e., of such a value λ_{cr} that for $\lambda > \lambda_{\text{cr}}$ self-ignition occurs, i.e., there is no solution of problem (8), while for $\lambda < \lambda_{\text{cr}}$ there is establishment of the process, i.e., there exists a solution of problem (8).

We shall assume $\lambda > 0$ and $F(0) > 0$ (the case $F(0) < 0$ can be considered analogously, and in the case $F(0) = 0$ there is always the trivial solution). Below we discuss only nonnegative solutions of problem (8). If, for example, $F(u) > 0$ for $u \leqslant 0$, then problem (8) can have only nonnegative solutions. The following facts are established without difficulty:

1. For solvability of problem (8) it is sufficient that

$$\max_{0 \leqslant u \leqslant \alpha} F(u) \leqslant \frac{2n\alpha}{\lambda} \tag{9}$$

for at least one α. From this one can obtain a lower estimate for λ_{cr}. Inequality (9) arises if one attempts to satisfy condition A by means of the function $v = \alpha(1 - r^2)$, $r^2 = \sum_i x_i^2$.

2. For solvability of problem (8) it is necessary that the equation

$$\lambda F(u) - \lambda_0 u = 0 \tag{10}$$

have a positive root. Here λ_0 is the first eigenvalue of the problem

$$\Delta u + \lambda u = 0, \quad u|_S = 0. \tag{11}$$

This follows from the lemma if one uses $v(x,t) = t u_0(x)$, where $u_0(x)$ is a suitably normalized eigenfunction of problem (10), corresponding to λ_0. From this one can obtain an upper estimate for λ_{cr}. If

$$\min_{u>0} \frac{F(u)}{u} = a > 0,$$

then $\lambda_{\text{cr}} \leqslant \dfrac{\lambda_0}{a}$.

3. If $F(u_0) = 0$ for some $u_0 \geqslant 0$, then problem (8) is solvable for any λ and $0 \leqslant u \leqslant u_0$. If, however, $F(u) > 0$ for $u \geqslant 0$, then for solvability for any λ it is sufficient that

$$\lim_{u \to \infty} \frac{F(u)}{u} = 0. \tag{12}$$

Indeed, (12) implies (9) for any λ. The condition $F(u_0) = 0$ for $u_0 \geqslant 0$ or $\lim\limits_{u \to \infty} \dfrac{F(u)}{u} = 0$ is also necessary for solvability of problem (8) for any λ.

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Note: Figure translations are in progress. See original paper for figures.