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L. D. ESKIN

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Abstract

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MATHEMATICS

L. D. ESKIN

THE ζ-FUNCTION ON THE GROUP OF UNITARY MATRICES

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Pleijel and Minakshisundaram ([1]), in connection with boundary-value problems for the operator $\Delta + \lambda$ (Δ is the Laplace–Beltrami operator of a Riemannian manifold), consider the Dirichlet series

$$\zeta(P, Q; s) = \sum_m \frac{U_m(P)U_m(Q)}{\lambda_m^s}. \tag{1}$$

In (1), $U_m(P)$ is a normalized eigenfunction of the boundary-value problem, belonging to the eigenvalue λ_m. The series (1) converges absolutely and uniformly if $\operatorname{Re} s$ is sufficiently large. In ([1]) it is proved that, for $P \neq Q$, $\zeta(P, Q; s)$ can be analytically continued to the whole complex plane as an entire function of s, whereas if $P = Q$, the analytic continuation is a meromorphic function of s with simple poles $N/2,\ N/2-1, \dots, 1$ ($N = 2k$ is the dimension of the manifold); $N/2,\ N/2-1, \dots$ ($N = 2k+1$). In ([2]) Minakshisundaram carried out the computation of the residues at the poles of $\zeta(P, P; s)$ on Euclidean space; in ([3,4]) the ζ-function (and, in essence, the residues at the poles) is computed on the real and complex spheres. In the present paper an analogous result (the computation of the residues) is obtained for the manifold of the group of n-dimensional unitary matrices $\mathfrak{A}_n$. In this case (as also in the case of the sphere), the role of boundary conditions is played by certain periodicity conditions with respect to the parameters of the manifold.

We shall consider on the group $\mathfrak{A}_n$ the heat equation

$$\frac{\partial w}{\partial t} = \Delta w,$$

where Δ is the Laplace–Beltrami operator for the metric $\operatorname{Sp}[(g^{-1}dg)^2]$. The radial part ([5]) of the operator Δ is easily computed and has the form

$$\overset{0}{\Delta} = \frac{1}{j(t)} \left[\sum_{k=1}^{n} \frac{\partial^2}{\partial t_k^2} + 1^2 + 2^2 + \cdots + (n-1)^2 \right] j(t),$$

$$j(t) = \prod_{s<r} (e^{it_s} - e^{it_r})\,; \qquad e^{it_1}, \dots, e^{it_n} \text{ are the eigenvalues of the matrix } g.$$

Let us construct the fundamental solution $w(g, g_1; t)$ of the Cauchy problem for the heat equation on the group $\mathfrak{A}_n$. Since the delta-function $\delta(g)$ is constant on each conjugacy class and the operator Δ commutes with shifts, by virtue of the uniqueness of the fundamental solution of the Cauchy problem, $w(g, g_1; t)$ depends only on the complex distance between the points g and g_1, i.e.

$$w(g, g_1; t) = w(t_1, \dots, t_n; t);$$

$e^{it_1}, \dots, e^{it_n}$ are the eigenvalues of the matrix $g_1^{-1} g$. The explicit form of the fundamental solution is given in Theorem 1.

Theorem 1. *The fundamental solution of the Cauchy problem for the heat equation on the group $\mathfrak{A}_n$ is given by the formula*

$$w(g, g_1; t) = \frac{(-i)^{n(n-1)/2}}{1!\,2! \cdots (n-1)!} \frac{e^{[1^2+2^2+\cdots+(n-1)^2]t}}{\prod_{s<r} (e^{it_s} - e^{it_r})} \times$$
$$\times \prod_{s<r} \left(\frac{\partial}{\partial t_s} - \frac{\partial}{\partial t_r} \right) \left\{ \vartheta_3 \left(\frac{t_1}{2\pi}; \frac{it}{\pi} \right) \vartheta_3 \left(\frac{t_2}{2\pi}; \frac{it}{\pi} \right) \cdots \vartheta_3 \left(\frac{t_n}{2\pi}; \frac{it}{\pi} \right) \right\}; \qquad (2)$$

$\theta_3(v; \tau)$ is the Jacobi theta-function:

$$\theta_3(v, \tau) = \sum_{m=-\infty}^{\infty} e^{(m^2\tau + 2mv)\pi i}, \qquad \operatorname{Im} \tau > 0.$$

Using the easily proved relation

$$\prod_{s<r} \left(\frac{\partial}{\partial t_s} - \frac{\partial}{\partial t_r} \right) e^{-(t_1^2+\cdots+t_n^2)} = (-2)^{n(n-1)/2} \prod_{s<r} (t_s - t_r) e^{-(t_1^2+\cdots+t_n^2)}$$

and the Jacobi transformation for theta-functions, one can transform (2) to a form with the singular part (for $g = g_1$) explicitly separated. As the singular part of the fundamental solution we obtain the expression

$$\frac{(i/2)^{n(n-1)/2} \pi^{n/2} e^{[1^2+\cdots+(n-1)^2]t}}{1!2! \cdots (n-1)!\, t^{n^2/2} \prod_{s<r} (e^{it_s} - e^{it_r})} \prod_{s<r} (t_s - t_r) e^{-(t_1^2+\cdots+t_n^2)/4t} \qquad (3)$$

(note that the real dimension of the group $\mathfrak{A}_n$ is equal to n^2). The functions $\zeta(g, g_1; s)$ and $w(g, g_1; t)$ are related to one another by the relation (for $\operatorname{Re} S$ sufficiently large)

$$\zeta(g, g_1; s) = \frac{1}{\Gamma(s)} \int_0^\infty [w(g, g_1; t) - 1]\, t^{s-1}\, dt; \tag{4}$$

$\Gamma(s)$ is Euler' s Γ-function. The analytic continuation of $\zeta(g, g_1; s)$ is constructed with the aid of (4). Formula (3) for the singular part and the investigation of the regular part of the fundamental solution (2) for $g = g_1$ make it possible to examine completely the singularities of the function $\zeta(g, g_1; s)$.

Theorem 2. The ζ-function $\zeta(g, g_1; s)$ on the group $\mathfrak{A}_n$ can be analytically continued to the whole plane of the complex variable s. This analytic continuation is an entire function of s with "trivial zeros" at the negative integral points when $g \neq g_1$. The function $\zeta(g, g; s)$ $(g = g_1)$ is a meromorphic function of s with simple poles

$$n^2/2,\ n^2/2 - 1, \dots, 1, \quad \text{if } n \text{ is even},$$

$$n^2/2,\ n^2/2 - 1, \dots, \quad \text{if } n \text{ is odd}.$$

The residue at the pole $s = n^2/2 - k$ is equal to

$$\frac{(1/2)^{n(n-1)/2} \pi^{n/2}}{1!2! \cdots (n-1)!\, \Gamma(n^2/2 - k)} \cdot \frac{[1^2 + \cdots + (n-1)^2]^k}{k!}.$$

Kazan State University
named after V. I. Ulyanov-Lenin

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