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Abstract

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MATHEMATICS

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QUADRATURE AND INTERPOLATION FORMULAS ON TENSOR PRODUCTS OF CERTAIN CLASSES OF FUNCTIONS

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Let K be some linear normed space and let $\tau_i, \tau_i^{(j)}$ be some of its elements. Let K^s be the tensor product of K with itself s times, i.e., the space of formal finite sums of the form

$$\sum_{1\le p\le N} \lambda_p \tau_p^{(1)} \otimes \cdots \otimes \tau_p^{(s)}$$

(where λ_p are numbers), for which addition and multiplication by numbers are defined in the trivial way, factored by all relations of the form

$$\left(\sum_{p=1}^{N_1} \lambda_p^{(1)} \tau_p^{(1)}\right) \otimes \cdots \otimes \left(\sum_{p=1}^{N_s} \lambda_p^{(s)} \tau_p^{(s)}\right) = \sum_{p_1=1}^{N_1} \cdots \sum_{p_s=1}^{N_s} \lambda_{p_1}^{(1)} \cdots \lambda_{p_s}^{(s)} \tau_{p_1}^{(1)} \otimes \cdots \otimes \tau_{p_s}^{(s)}.$$

We shall consider such K^s for which one can introduce a norm so that

$$\|\tau_1 \otimes \cdots \otimes \tau_s\| = \|\tau_1\| \cdots \|\tau_s\|.$$

Then the following theorem holds.

Theorem. Let ϑ_ν, $(\nu = 0, 1, \dots, q)$, and I be such elements of K that, for $\alpha > 0$,

$$\|I\| \le B, \qquad \|\vartheta_\nu\| \le B, \qquad \|I - \vartheta_\nu\| \le A \cdot 2^{-\nu\alpha},$$
$$\theta_0 = \vartheta_0, \qquad \theta_\nu = \vartheta_\nu - \vartheta_{\nu-1} \quad (\nu \ge 1).$$

Then

$$\left\| I \otimes \cdots \otimes I - \sum_{\nu_1+\cdots+\nu_s \le q} \theta_{\nu_1} \otimes \cdots \otimes \theta_{\nu_s} \right\| \le C(A, B, s, \alpha) \frac{q^{s-1}}{2^{\alpha q}}.$$

Moreover, the written sum is in fact a linear combination of terms $\vartheta_{\nu_1} \otimes \cdots \otimes \vartheta_{\nu_s}$, with $q - s \le \nu_1 + \cdots + \nu_s \le q$.

The proof of this theorem is completely trivial in the case when K and K^s are spaces of numbers, the operation of tensor multiplication coincides with ordinary multiplication, and the norm of a number is understood as its modulus. In the general case the proof is carried out in a completely analogous way.

From this simple theorem one can derive a number of interesting consequences concerning quadrature and interpolation formulas on certain classes of functions, for example $W_s^\alpha(1)$, $E_s^\alpha(2)$, $H_s^\alpha(^3, ^4)$.

Let, for example, K be the space of linear continuous functionals on the class $W_1^\alpha(1)$—the class of 1-periodic functions expandable in a Fourier series

$$f(x) = \sum_{-\infty}^{\infty} a_m e^{2\pi i m x}$$

with norm

$$\|f\|^2 = \sum_{-\infty}^{\infty} |c_m|^2 \bar{m}^{2\alpha}.^*$$

Then the elements of K can be identified with sequences

$$\tau = (\dots, c_m, \dots),$$

where

$$\|\tau\|^2 = \sum_{-\infty}^{\infty} |c_m|^2 \bar{m}^{-2\alpha}.$$

Let

$$\tau^{(i)} = (\dots, c_m^{(i)}, \dots).$$

* Here and below $\bar{m} = \max(1, |m|)$.

Define $\tau^{(1)} \otimes \cdots \otimes \tau^{(s)}$ as an infinite tensor of rank s

$$T = \begin{pmatrix} & \vdots & \\ \cdots c_{m_1 \dots m_s} \cdots & & \\ & \vdots & \end{pmatrix}, \tag{1}$$

where $c_{m_1 \dots m_s} = c_{m_1}^{(1)} \cdots c_{m_s}^{(s)}$. Addition of tensors and their multiplication by numbers will be understood in the usual sense, and the norm of the tensor (1) will be defined by the formula

$$\|T\|^2 = \sum_{m_1, \dots, m_s} |c_{m_1 \dots m_s}|^2 (\bar{m}_1 \cdots \bar{m}_s)^{-2\alpha}.$$

Then K^s will be, generally speaking, a subspace of the linear normed space of linear continuous functionals (l.c.f.) on the class W_s^α, and the norms in both classes will coincide. Let us take, as I, the l.c.f. of integration on W_1^α: $(I, f(x)) = \int_0^1 f(x)\,dx$. Then $I \otimes \dots \otimes I$ will be the l.c.f. of s-fold integration in W_s^α. As ϑ_ν take the l.c.f. corresponding to some good quadrature formula in W_1^α with 2^ν nodes of integration, for example such that:

$$(\vartheta_\nu, f(x)) = \frac{1}{2^\nu} \sum_{k=1}^{2^\nu} f\left(\frac{k}{2^\nu}\right).$$

It is easily verified that for $\alpha > 1/2$, $\|I\| = 1$, $\|\vartheta_\nu\| \le B(\alpha)$,

$$\|I - \vartheta_\nu\| \le A(\alpha) \cdot 2^{-\nu\alpha}.$$

To clarify the meaning of the assertion of the theorem in this case, let us make the following remark. If $\delta(\xi) \in K$ is defined by the equality $(\delta(\xi), f(x)) = f(\xi)$, then this l.c.f. is identified with the sequence $(\dots, e^{2\pi i m\xi}, \dots)$, and then the l.c.f. $\delta(\xi_1, \dots, \xi_s) = \delta(\xi_1) \otimes \dots \otimes \delta(\xi_s)$ is identified with the tensor (1), where $c_{m_1 \dots m_s} = e^{2\pi i(m_1\xi_1 + \dots + m_s\xi_s)}$, and therefore for it

$$(\delta(\xi_1, \dots, \xi_s), f(x_1, \dots, x_s)) = f(\xi_1, \dots, \xi_s)$$

for $f \in W_s^\alpha$. If $\tau^{(i)}$ is a linear combination of $\delta(\xi_1^{(i)}), \dots, \delta(\xi_{N_i}^{(i)})$, then $\tau^{(1)} \otimes \dots \otimes \tau^{(s)}$ is a linear combination of $\delta(\xi_{n_1}^{(1)}, \dots, \xi_{n_s}^{(s)})$ $(1 \le n_i \le N_i)$. Since in our case ϑ_ν, and hence also θ_ν, are linear combinations of

$$\delta\left(\frac{1}{2^\nu}\right), \dots, \delta\left(\frac{2^\nu}{2^\nu}\right),$$

it follows that $\theta_{\nu_1} \otimes \dots \otimes \theta_{\nu_s}$ is a linear combination of the terms

$$\delta\left(\frac{n_1}{2^{\nu_1}}, \dots, \frac{n_s}{2^{\nu_s}}\right).$$

Consequently, the l.c.f.

$$\sum_{\nu_1 + \dots + \nu_s \le q} \theta_{\nu_1} \otimes \dots \otimes \theta_{\nu_s},$$

when applied to any function from W_s^α, will give a linear combination of its values at the grid nodes

$$\left(\frac{n_1}{2^{\nu_1}}, \dots, \frac{n_s}{2^{\nu_s}}\right), \quad 1 \le n_i \le 2^{\nu_i}, \quad \nu_1 + \dots + \nu_s \le q, \tag{2}$$

$$\left(\sum_{\nu_1+\cdots+\nu_s \le q} \theta_{\nu_1} \otimes \cdots \otimes \theta_{\nu_s}, f(x_1, \dots, x_s) \right) =$$

$$= \sum_{\nu_1+\cdots+\nu_s \le q} \sum_{1 \le n_i \le 2^{\nu_i}} p_{\nu_1 \dots \nu_s}^{n_1 \dots n_s} f\left(\frac{n_1}{2^{\nu_1}}, \dots, \frac{n_s}{2^{\nu_s}} \right).$$

The coefficients $p_{\nu_1 \dots \nu_s}^{n_1 \dots n_s}$ can each be computed in $C(s)$ operations; moreover, by the theorem, the coefficients with $\nu_1 + \cdots + \nu_s < q - s$ will be equal to zero. It is easy to see that in this case we shall use the values-

tion of the function at $O(q^{s-1} \cdot 2^q)$ points. By virtue of our theorem and by the definition of the norm of a linear functional,

$$\left| \int_0^1 \cdots \int_0^1 f(x_1, \dots, x_s)\, dx_1 \cdots dx_s - \sum_{\nu_1+\cdots+\nu_s \le q} \sum_{1 \le n_i \le 2^{\nu_i}} p_{\nu_1 \dots \nu_s}^{n_1 \dots n_s} f\left(\frac{n_1}{2^{\nu_1}}, \dots, \frac{n_s}{2^{\nu_s}} \right) \right| \le$$

$$\le C(\alpha, s) \frac{q^{s-1}}{2^{\alpha q}} \|f\|_{W_s^\alpha}.$$

If the number of integration nodes is denoted by N, then $N = O(q^{s-1} \cdot 2^q)$, and the right-hand side of the last inequality will be

$$O\left(N^{-\alpha} \log^{(\alpha+1)(s-1)} N \right).$$

Taking, instead of W_1^α, the class E_1^α, i.e. defining in K the norm

$$\|\tau\| = \sum_{-\infty}^{\infty} |c_m|\, \bar{m}^{-\alpha},$$

we analogously obtain that the same quadrature formula gives the same order of error also on the class E_s^α. Taking, instead of W_1^α, the class E_2^α, instead of I –the linear functional of double integration over the unit square, and putting

$$\vartheta_\nu = \frac{1}{u_\nu} \sum_{k=1}^{u_\nu} \delta \left(\frac{k}{u_\nu}, \frac{k u_{\nu-2}}{u_\nu} \right) \qquad (u_1 = 1,\ u_2 = 2, \dots,\ u_n = u_{n-1} + u_{n-2})$$

and using the result of the work ([5]) in the form

$$\|I - \vartheta_\nu\| \le A(\alpha) \nu / u_\nu^\alpha,$$

we analogously arrive at a quadrature formula on E_{2s}^α with $O(q^{s-1}u_q)$ nodes, whose error, as can be seen by slightly modifying the proof of the main theorem, will be

$$O\left(\frac{q^{2s-1}}{u_q^\alpha}\right)$$

or, recalculated in terms of the number of nodes,

$$O\left(\frac{\log^{(s-1)(\alpha+2)+1} N}{N^\alpha}\right).$$

By modifying the proof a little further, one can obtain an estimate of the error of the quadrature formula also on the class E_{2s-1}^α in the form

$$O\left(\frac{\log^{(s-1)(\alpha+2)} N}{N^\alpha}\right).$$

For $\alpha \geq 2$ these results are more accurate than the estimates of errors of quadrature formulas over parallelepipedal grids obtained in the works ([2],[5]).

Let us now take in the theorem

$$I = \delta(\xi), \qquad \vartheta_\nu = \frac{1}{2^\nu}\sum_{k=1}^{2^\nu} V_{2^\nu+1}\left(\xi - \frac{k}{2^\nu}\right)\delta\left(\frac{k}{2^\nu}\right),$$

where

$$V_n(x) = \frac{\sin^2 2\pi nx - \sin^2 \pi nx}{n \sin^2 \pi x}$$

is the Vallée-Poussin operator ([6]). Using the boundedness of the integral and interpolation norm of V_n ([7]) and estimates of best approximations by trigonometric polynomials of periodic functions from the classes E_1^α, W_1^α, and $\widetilde{H}_1^\alpha$ (the last class in ([7]) is denoted by $W_*^{(r)}H^{(\beta)}$, where $r = [\alpha]$, $\beta = \{\alpha\}$), one can obtain that

$$\|I - \vartheta_\nu\|_{E_1^\alpha} = O(2^{-\nu(\alpha-1)}), \qquad \|I - \vartheta_\nu\|_{W_1^\alpha} = O(2^{-\nu(\alpha-1/2)}), \qquad \|I - \vartheta_\nu\|_{\widetilde{H}_1^\alpha} = O(2^{-\nu\alpha}).$$

Then from our theorem we obtain an interpolation formula with nodes (2), which gives accuracy (in the sense of the maximum error modulus) on E_s^α

$$O\left(\frac{\log^{\alpha(s-1)} N}{N^{\alpha-1}}\right),$$

on W_s^α

$$O\left(\frac{\log^{(\alpha+1/2)(s-1)} N}{N^{\alpha-1/2}}\right)$$

and on $\widetilde{H}_s^\alpha$

$$O\left(\frac{\log^{(\alpha+1)(s-1)} N}{N^{\alpha}}\right).$$

On the other hand, it can be shown that no interpolation formula with N nodes can give, on E_s^α, an accuracy better than

$$O\left(\frac{1}{N^{\alpha-1}}\right),$$

on W_s^α—better than

$$O\left(\frac{1}{N^{\alpha-1/2}}\right),$$

and on the class $\widetilde{H}_s^\alpha$—better than

$$O\left(\frac{1}{N^{\alpha}}\right).$$

Sharper lower estimates have at present been obtained by I. F. Sharygin ([13]).

In works ([1,4,9,10]) interpolation on parallelepipedal grids was considered, but, as is shown in ([1,10]), the best estimate in this case for all the classes considered cannot be better than $O\left(\frac{1}{N^{\alpha/2}}\right)$, so that the present result is the sharpest of the existing ones and can be improved only by logarithmic factors.

In the case of classes of nonperiodic functions H_s^α ([3,4]) or $W^{(r)}H^{(\beta)}$ ([7]), instead of the l.f. ϑ_ν considered one must take others, corresponding to good quadrature or interpolation formulas on these classes, for example Sard' s or Nikol' skii' s quadrature formulas ([8]), or Newton interpolation formulas. The error estimates in the nonperiodic case will be of the same order as in the periodic case (of course, for the corresponding classes), and will be more accurate than for the quadrature and interpolation formulas considered in ([3,4]).

The proposed construction is applicable to estimates of the diameters of the classes $\widetilde{H}_s^\alpha$ and of classes close to them. However, the method indicated in the paper yields only another proof of K. I. Babenko' s theorem ([11]). On the basis of the quadrature and interpolation formulas described one can also construct methods for the numerical integration of integral and differential equations, etc. Random elements K may be taken as the ϑ_ν, which makes it possible to obtain estimates in the mean for the remainder terms of quadrature formulas under random selection either of the nodes of integration themselves or of the number of these nodes, if the same results are known in the one-dimensional case. The results obtained differ substantially from the results of ([12]); for example, the mathematical expectation of the modulus of the error of a quadrature formula on W_s^α for $\alpha > 1/2$ will be

$$O\left(\frac{\log^{(\alpha+1/2)(s-1)} N}{N^{\alpha+1/2}}\right),$$

whereas in ([12]) it is estimated as

$$O\left(\frac{\log^{s+1} N}{N^{\alpha+1/2}}\right).$$

The applicability of our method to the classes $E_s^\alpha, W_s^\alpha, H_s^\alpha$ is due to the fact that the spaces of l.f. over them contain tensor products of spaces of l.f. over the corresponding one-dimensional spaces.

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Note: Figure translations are in progress. See original paper for figures.