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Abstract

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ON A CLASS OF SPACES CONTAINING ALL METRIC AND ALL LOCALLY BICOMPACT SPACES

(Presented by Academician P. S. Aleksandrov on February 7, 1963)

We shall give only the definitions and results. All spaces in what follows are assumed to be completely regular.

§ 1. **Definition 1.** A system B of bicompact subsets of a space X is called a k-**base** of this space if, for an arbitrary bicompactum $F \subseteq X$, there is an element $\Phi \in B$ containing F. Obviously, a k-base exists in every space; moreover, bicompacta are characterized by the fact that they have a k-base consisting of only one element.

Definition 2. The k**-weight** of a space is the minimum of the cardinalities of all k-bases of this space.

The concept of a k-base allows us to single out the following class of spaces, important for us:

Definition 3. We shall call a space X a **space of countable type** if it has a k-base consisting of bicompacta of countable character* (a k-base of countable type).

Lemma 1. *Let X be a space of countable type, F an arbitrary bicompactum in it, and U any neighborhood of it. Then there exists a bicompactum Φ of countable character such that $F \subseteq \Phi \subseteq U$.*

Turning to the study of the properties of spaces of countable type, we note that almost all results will be formulated for a somewhat broader class of spaces, which we shall call spaces of point-countable type and for which we assume only that each of their points is contained in some bicompactum of countable character (in other words, there exists a covering of the space by bicompacta of countable character).

The spaces of countable type include all metric spaces and, on the other hand, all locally bicompact spaces and even all spaces complete in the sense of Čech. However, not every space has a k-base of countable type, as is shown, in particular, by:

Theorem 1. *An arbitrary space of point-countable type is a k-space.*

Under closed mappings the property of being a space of countable type need not be preserved; however, the following is true:

Theorem 2. *Preimages and images of spaces of countable type under perfect mappings are spaces of countable type.*

Theorem 3. *The product of a countable set of spaces of countable (point-countable) type is a space of countable (point-countable) type.*

Theorem 3, which is proved simply, is of some interest, since together with Theorem 1 it shows that the class of spaces of countable type forms a subclass of k-spaces closed under countable products—this is important, since the whole class of k-spaces is not closed even under finite products (Dowker' s well-known example).

* A bicompactum $F \subseteq X$ is called a bicompactum of **countable character** (bX) if there exists a countable system of open sets $\varphi = \{U_n\}$ such that, for every open $U \supseteq F$, there is a $U_n \in \varphi$ for which $F \subseteq U_n \subseteq U$.

Theorem 4. *Let $f : X \to Y$ be an open-closed mapping of a weakly paracompact space X onto a space Y of point-countable type. Suppose, moreover, that Y is nowhere locally bicompact. Then the mapping f is bicompact.*

Theorem 4, in the case when X and Y are metric spaces, was proved earlier by A. D. Taimanov.

Theorem 5. *Let $f : X \to Y$ be a closed mapping of a metric space X onto a space Y of point-countable type. Then Y is metrizable.*

Theorem 6. *Spaces of point-countable type, and only they, are continuous, open, Y-bicompact images of metric spaces.**

Corollary. *Every topological space complete in the sense of Čech is a continuous open Y-bicompact image of some metric space.*

Remark. Theorem 6 can also be applied to the characterization of spaces of countable type—one must additionally require that for every bicompactum $F \subseteq Y$ there be a point $x \in X$ such that $fx \supseteq F$.

§ 2. p-Spaces

A. Definition 4. Let X and Y be topological spaces and $Y \subset X$. We shall say that Y **belongs to the class** $p(X)$ (i.e. $Y \in p(X)$) if there exists a countable system $\varphi = \{\gamma_i\}$ of open coverings of the space Y such that for an arbitrary point $y \in Y$ the relation

$$\bigcap_{i=1}^{\infty} \gamma_i y \subseteq Y$$

holds. Here by $\gamma_i y$ is meant the star of the point y with respect to the system γ_i, i.e. the union of all elements of γ_i containing the point y.

Definition 5. A space X is called a ***p*-space** if X belongs to the class $p(\beta X)$, where βX is the Čech extension of the space X.

The scope of the class of spaces introduced by Definition 5 is estimated from below by the following two theorems:

Theorem 7. *Every metric space is a p-space.*

Theorem 8. *Every Borel space of countable class (see ([5])) is a p-space.*

Corollary. *Spaces complete in the sense of Čech are p-spaces (and, consequently, k-spaces; see Theorem 9).*

An upper estimate is given by

Theorem 9. *Every p-space is a space of point-countable type.*

Corollary. *Every p-space is a k-space.*

The main theorems of this subsection, alongside Theorems 7 and 8, include Theorem 10:

Theorem 10 (additive). *Let $X = \bigcup_{\alpha<\tau} X_\alpha$ be a space X represented as the sum of τ pieces (where τ is a cardinality) of its subspaces, the weight of each of which also does not exceed τ. Then the weight of the space X also does not exceed τ.*

There exists a simple example showing that this need not be so for non-p-spaces.

Corollary. *The weight of a p-space does not exceed its cardinality**.*

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* The theory of multivalued mappings has recently been developed in the works of V. I. Ponomarev. We use the terminology from ([8]). V. I. Ponomarev proved ([7]) that the single-valued open continuous images of metric spaces are precisely the spaces with the first axiom of countability.

** This is a strengthening of a result of P. S. Aleksandrov on bicompacts.

A strengthening of P. S. Aleksandrov's theorem that under continuous mappings the weight of bicompacta does not increase ([3]) is

Theorem 11. *Under a continuous mapping of an arbitrary space onto a p-space, the weight of the image cannot be greater than the weight of the preimage.*

Corollary. *Let $f : X \to Y$, where X is a metrizable space with a countable base, f is a continuous mapping, and Y is a p-space; then Y is also a metrizable space with a countable base.*

For completeness we give the following result:

Theorem 12. *If the space X belongs to the class $p(B)$, where B is a bicompactum, and the weight of the space X does not exceed τ, then X has in B an external base of cardinality $\leqslant \tau$.*

In connection with Theorems 9, 10, 11, and 12 one may point to the papers ([4–6]), where, in particular, the definition is given of the concept of a net of a space, on which the proof of these theorems is based.

Theorem 13. *The preimage of a p-space under a perfect mapping is a p-space.*

Corollary. *The absolute* ([9]) *of a p-space is a p-space.*

B. We have defined p-spaces as spaces of class p in their Čech extension. However, it turns out that the property of a space of belonging to the class p, like the property of being a G_δ, is invariant with respect to its bicompact extensions. More precisely, the following holds:

Theorem 14. *The following assertions concerning a space X are equivalent: a) $X \in p(\beta X)$; b) $X \in p(bX)$ for some bicompact extension bX of the space X; c) $X \in p(bX)$ for an arbitrary bicompact extension bX of the space X.*

With the aid of Theorem 12 one obtains in a natural way:

Theorem 15. *The product of a countable set of p-spaces is a p-space.*

§ 3. A. Paracompact p-spaces.

The main theorem of this subsection is

Theorem 16. *A topological space is perfectly mapped onto some metric space $*$ if and only if it is a paracompact p-space $**$.*

Corollary 1. *The product of a countable set of paracompact p-spaces is a paracompact p-space.*

Theorem 16 and Corollary 1 from it strengthen the following results of Z. Frolík: 1) complete paracompact spaces are perfectly mapped onto metric spaces, and 2) the product of complete paracompact spaces is a paracompact space ([10]).

A partial solution of the well-known problem of Michael $***$: whether the product of a paracompactum with a metric space is necessarily a paracompactum, gives

Corollary 2. *The product of a paracompact p-space with a metric space is a paracompact p-space.*

Theorem 17. *The product of a paracompact p-space with a perfectly normal paracompact space is a paracompact space.*

Paracompact p-spaces are widely distributed, as the following two theorems show.

Theorem 18. *Let $X \subseteq Y$, where X is a paracompactum and Y is a p-space. Then there exists a paracompact p-space Z such that $X \subseteq Z \subseteq Y$.*

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* The problem of characterizing perfect preimages of metric spaces was also considered (independently of me) by V. I. Ponomarev, and he obtained related results.

** Perfect preimages of spaces with a countable base are functionally bicompact p-spaces (for them the γ_i in Definition 4 may be taken to be countable).

*** I have just learned from a note kindly sent to me by Prof. Michael that he has succeeded in solving this problem negatively, by constructing a very attractive example.

Let us note that Z. Frolik and V. I. Ponomarev also studied the problem of approximating some spaces by others.

Theorem 19. Let $f : X \to Y$, where f is a continuous mapping of a normal space X onto a metric space Y. Then there exists a paracompact p-space $\widetilde{X} \supset X$ such that f extends to a mapping $\tilde{f} : \widetilde{X} \to Y$ ($\tilde{f}x = fx$ for $x \in X$), and the mapping $\tilde{f}$ is perfect.

B. Theorem 20. A space that is a closed image of a metric space is metrizable if and only if it is a p-space.

Theorem 21. Suppose that

$$X \xrightarrow{f} Y \xrightarrow{\varphi} Z,$$

where X and Z are metric spaces, f is a closed mapping, and φ is perfect. Then the space Y is metrizable.

Remark. Here it is not enough to assume that the mapping φ is merely closed.

We see that nonmetrizable paracompacts which are closed images of metric spaces are already so "exhausted" that they cannot be mapped perfectly onto any metric space; and, on the other hand, if a nonmetrizable space is mapped perfectly onto a metric space, then it is so "bad" that no metric space can be mapped closedly onto it.

Theorems 20 and 21 answer one of the questions contained in P. S. Aleksandrov' s survey lecture at the symposium in Prague in 1961.

B. Characteristics of paracompact p-spaces. We have been able to see that among paracompacts the paracompact p-spaces possess especially remarkable properties. A natural question is: when is a paracompact a p-space? The preceding discussion provides two characterizations: the first—given in the definition of a p-space—through the position of the paracompact in its arbitrary extension, and the second—as a perfect preimage of a metric space.

But both of these characterizations are external; we shall now give an internal criterion.

Definition 6. A system D of open sets of a space X is called **regular on a set** $M \subseteq X$ if, for an arbitrary neighborhood U of the set M, the following holds: 1) for every point $x \in M$ there exists $V \in D$, $x \in V \subseteq U$; 2) the set of elements of the system D that intersect simultaneously both M and $X \setminus U$ is empty or finite.

Theorem 22. A paracompact X is a p-space if and only if there exist a cover of it by bicompacts $\pi = \{\Phi_\alpha\}$ and a system of open sets D, regular on each element of the cover π.

Remark. If the system π in the formulation of Theorem 20 consists of all singleton subsets of the space X, then D automatically turns out to be a uniform base of this space. Then X is metrizable by P. S. Aleksandrov' s theorem ([2]).

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CITED LITERATURE

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[4] **A. Arkhangel' skii**, DAN, **126**, No. 2, 239 (1959).
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Note: Figure translations are in progress. See original paper for figures.