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1963

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Abstract

Full Text

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Existence of Solutions of Linear Hamiltonian Equations with Unbounded Operator Coefficients

(Presented by Academician V. I. Smirnov on 4 III 1963)

1°. Consider the Hamilton equation

$$J\frac{dx}{dt} = H(t)x \qquad (0 \leq t \leq T) \tag{1}$$

in a complete separable complex Hilbert space $\mathfrak{H}$. Here J is a bounded operator with a bounded inverse and is skew-Hermitian ($J^* = -J$); $H(t)$ is a self-adjoint, positive definite operator

$$H^*(t) = H(t) \geq \beta I \quad (\beta = \text{const} > 0).$$

Equation (1) occurs frequently in applications (^{1}).

Differential equations in abstract spaces have been studied by T. Kato, M. I. Vishik, O. A. Ladyzhenskaya, M. A. Krasnosel' skii, and other authors (see the survey article by Yu. L. Daletskii (^{2})). In contrast to the first-order equations considered in most works (^{2}), the resolving operator for equation (1) is unbounded. Certain second-order equations considered by O. A. Ladyzhenskaya and M. I. Vishik ($^{3-5}$) reduce to equation (1). For these equations results are obtained (see Theorems 5 and 6) that are close to the results of ($^{3-5}$).

Below it is assumed that the following smoothness condition on the operator $H(t)$ is satisfied:

A. The positive root $H^{1/2}(t)$ of the operator $H(t)$ has a constant domain of definition $D(H^{1/2})$, and for $t, t' \in [0, T]$, for any elements $x, y \in D(H^{1/2})$, the estimate

$$\left|(H^{1/2}(t)x, H^{1/2}(t)y) - (H^{1/2}(t')x, H^{1/2}(t')y)\right| \leq$$

$$\leq \text{const}\,|t - t'|\,\|H^{1/2}(t)x\|\,\|H^{1/2}(t)y\|$$

holds.

We define a generalized solution of equation (1), following S. L. Sobolev, with the natural modifications here.

Definition 1. A **generalized solution of equation** (1) is any weakly continuous function $x(t)$ having the following properties:

1) $x(t) \in D(H^{1/2})$ for all $t \in [0, T]$ and $\|H^{1/2}(0)x(t)\| \leq \text{const}$;
2) for every $t \in [0, T]$ the integral identity

$$(Jx(t), y) - (Jx(0), y) = \int_0^t (H(s)^{1/2}x(s), H(s)^{1/2}y)\, ds \tag{2}$$

is satisfied for any element $y \in D(H^{1/2})$.

Formally, identity (2) is obtained by multiplying equation (1) by the element y and integrating.

Theorem 1. *Equation (1) with the initial condition $x(0) \in D(H^{1/2})$ has a unique generalized solution. For any two generalized solutions $x(t)$ and $y(t)$, the relation*

$$(Jx(t), y(t)) = \text{const} \tag{3}$$

holds.

Theorem 2. The resolving operator $X(t)$, defined on $D(H^{1/2})$ by the generalized solution $x(t)$ of equation (1) by the formula $X(t)x(0) = x(t)$, maps the set $D(H^{1/2})$ one-to-one onto itself, and for it the following representation, convergent in the strong sense on $D(H^{1/2})$, is valid:

$$X(t) = \lim_{n\to\infty} \prod_{j=0}^{n} H(t_j)^{-1/2} e^{iK_j \Delta t} H(t_j)^{1/2},$$

where

$$K_j = iH(t_j)^{1/2} J H(t_j)^{1/2}$$

is a self-adjoint operator and $t_j = j\Delta t$, $\Delta t = t/n$. The operator $X(t)$ has the form

$$X(t) = H(0)^{-1/2} Z(t) H(0)^{1/2},$$

where $Z(t)$ is uniformly bounded in $t \in [0, T]$ together with its inverse.

Theorem 3. The generalized solution $x(t)$ has the following properties:

1) $H^{1/2}(0)x(t)$ is a weakly continuous function of t;
2) for almost all $t \in [0, T]$ the function $H^{-1/2}(0)Jx(t)$ has a strong derivative and

$$\frac{d}{dt}\left[H^{-1/2}(0)Jx(t)\right] = \left[H^{1/2}(t)H^{-1/2}(0)\right]^* H^{1/2}(t)x(t); \tag{4}$$

3) $x(t)$ assumes the initial value $x(0)$ in the sense of weak continuity.

Properties 1), 2), 3) of the function $x(t)$ may be taken as the basis for the definition of a generalized solution of equation (1).

We outline the proof. Divide the interval $[0, T]$ into n equal parts by the points

$$t_j = \frac{T}{n} j \quad (j = 0, 1, 2, \dots, n).$$

On each interval $[t_j, t_{j+1}]$ $(j = 0, 1, \dots, n-1)$ in equation (1) replace the operator $H(t)$ by

$$H_j = H(t_j)$$

and solve the resulting equation successively on the intervals. The generalized solution of the equation $J\dot{x} = H_j(x)$ on the interval $[t_j, t_{j+1}]$ is given by the formula

$$x_n^{(j)}(t) = H_j^{-1/2} e^{iK_j(t-t_j)} H_j^{1/2} x_n^{(j)}(t_j).$$

The functions

$$x_n(t) = x_n^{(j)}(t) \quad (t \in [t_j, t_{j+1}];\ j = 0, 1, 2, \dots, n-1)$$

by virtue of condition A possess the property

$$\|H^{1/2}(0) x_n(t)\| \leq C \|H^{1/2}(0) x(0)\|, \tag{5}$$

where the constant C does not depend on n or on $x(0)$. From the sequence $x_n(t)$ one can choose a subsequence weakly converging to some function $x(t)$. The function $x(t)$ is defined for almost all t, satisfies identity (2), and, by virtue of (5),

$$\|H^{1/2}(0) x(t)\| \leq \text{const}\, \|H^{1/2}(0) x(0)\|. \tag{6}$$

Identity (2), by weak continuity, extends $x(t)$ to all t, and hence all the remaining properties mentioned in Definition 1 follow. Differentiating identity (2) with respect to t, after simple arguments we obtain equation (4), by means of which relation (3) is easily justified.

It follows from what has been proved that equation (1) can be integrated in the negative direction of the t-axis. From this fact and relation (3) the uniqueness of the generalized solution follows at once. From estimate (6) and uniqueness, the remaining assertions of Theorem 2 are obtained in a simple way.

Assertion 2) of Theorem 3 was proved above; assertions 1), 3) of Theorem 3 follow from the weak continuity of $x(t)$ and estimate (6).

2°. Let the operators J and $H(t)$ be subject to the additional condition:

B. $D(H^{1/2}) \supset R(J^{-1}H(t))$, the operator

$$\Gamma(t) = H^{1/2}(t) J^{-1} H(t)$$

has a constant domain of definition $D(\Gamma)$, dense in $\mathfrak{H}$, and for $t, t' \in [0, T]$ one has

$$\|\Gamma(t)\Gamma^{-1}(t')\| \leq 1 + |t - t'| \operatorname{const}.$$

Then the generalized solution of equation (1) becomes an ordinary solution if $x(0) \in D(\Gamma)$, as the following theorem shows:

Theorem 4. If conditions A and B are satisfied, then there exists a unique function $x(t)$ possessing the following properties:

1) $x(t) \in D(\Gamma)$ for all $t \in [0, T]$ and $\|\Gamma(0)x(t)\| \leq \operatorname{const} \|\Gamma(0)x(0)\|$.
2) The function $H^{1/2}(0)x(t)$ is strongly continuous and for almost all t has a strong derivative, which is weakly continuous.
3) For almost all t the relation

$$\frac{d}{dt}\,[H^{1/2}(0)x(t)] = H^{1/2}(0)J^{-1}H(t)x(t).$$

The requirement of positive definiteness of the operator $H(t)$ under an additional boundedness assumption can be replaced by the requirement of semiboundedness. If, for example, the operator $H_1(t) = H(t) + \gamma I$ (where γ is some positive constant) satisfies condition A and the operator

$$H_1^{1/2}(t)J^{-1}H_1^{-1/2}(t')$$

is meaningful and bounded for $t, t' \in [0, T]$, then equation (1) with the initial condition $x(0) \in D(H_1^{1/2})$ has a unique generalized solution. This generalized solution has the properties listed in Theorems 1, 2, 3, with the operator $H(t)$ replaced in the statements of those theorems by $H_1(t)$.

3°. In the space $\mathfrak{H}$ the hyperbolic equation

$$\frac{d^2u}{dt^2} + P(t)u = 0, \tag{7}$$

where $P^*(t) = P(t) \geq \beta I$ $(\beta > 0)$, by the substitution given for the finite-dimensional case, for example in $^{(6)}$, reduces to equation (1). The existence of a solution of the resulting equation is equivalent to the existence of a solution of equation (7) in the sense of the following definition.

Definition 2. We shall call a strongly continuous function $u(t)$ possessing the following properties a **generalized solution of equation** (7):

1) for all $t \in [0, T]$, $u(t) \in D(P^{1/2})$, and $P^{1/2}(0)u(t)$ is weakly continuous;
2) $u(t)$, by continuity at $t = 0$, assumes the initial value $u(0)$ and has a weakly weakly continuous derivative $\dot{u}(t)$;

3) for any $t \in [0,T]$ and any element $y \in D(P^{1/2})$ the integral identity

$$\int_0^t (P^{1/2}(s)u(s),\ P^{1/2}(s)y)\,ds - (u(0),y) + (u(t),y) = 0$$

holds.

Suppose that the operator $P(t)$ satisfies condition C. $P(t)$ is subject to condition A with the operator $H(t)$ replaced in it by $P(t)$.

Theorem 5. Under condition C, equation (7) with initial conditions $u(0) \in D(P^{1/2})$ and $\dot{u}(0) \in \mathfrak{H}$ has a unique generalized solution.

When the initial conditions and the smoothness conditions on $P(t)$ are strengthened, the generalized solution of equation (7) becomes an ordinary solution, as the following theorem shows.

Theorem 6. If $u(0) \in D(P(0))$, $\dot{u}(0) \in D(P^{1/2})$, condition C is satisfied, and also $D(P(t)) = D(P(0))$ and

$$\|P(t)P^{-1}(t')\| \le 1 + (t-t')\,\mathrm{const}$$

for $t, t' \in [0,T]$, then there exists a unique function $u(t)$, which possesses the following properties:

1) $u(t)$ has a strong derivative $\dot{u}(t)$, for all $t \in [0,T]$,

$$u(t) \in D(P(0)), \qquad \dot{u}(t) \in D(P^{1/2}), \qquad \|P(0)u(t)\| \le \mathrm{const},$$

$$\|P^{1/2}(0)\dot{u}(t)\| \le \mathrm{const};$$

2) $u(t)$, for almost all t, has a second strong derivative and, for these t, satisfies equation (6);

3) for $t = 0$, $u(t)$ and $\dot{u}(t)$, by continuity, take the values $u(0)$ and $\dot{u}(0)$, respectively.

Theorems 5 and 6 follow respectively from Theorems 1 and 4.

4°. Analogous theorems are valid for the equation

$$\frac{d}{dt}M(t)\frac{du}{dt} + iQ\frac{du}{dt} + P(t)u = 0$$

with the "gyroscopic term" $iQ\,du/dt$, where $M(t), P(t), Q$ are self-adjoint operators, $M(t) \ge \beta I,\ P(t) \ge \beta I,\ \beta > 0$, which reduces (6) to equation (1).

Received
25 II 1963

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Note: Figure translations are in progress. See original paper for figures.