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Abstract

Full Text

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ON FINITELY MULTIPLE OPEN MAPPINGS OF MANIFOLDS

(Presented by Academician P. S. Aleksandrov, 25 I 1963)

1. Here an outline is given of the proof of several propositions on finitely multiple open mappings of manifolds. Theorems 2 and 3 formulated below generalize the known theorems of Newman and Smith on periodic transformations ([1]), while Theorem 1 generalizes Montgomery' s theorem ([6]).

Open-and-closed continuous mappings correspond to completely continuous decompositions into closed sets (see, for example, ([2])). Let $f : X \to Y$ be such a mapping and let X be locally connected. We shall constantly use the following consideration. If x is an isolated point of the element of the decomposition containing it, then there is a connected neighborhood Ux on which f induces a completely continuous decomposition.

A point x is called a point of local homeomorphism if it has a neighborhood with which the elements of the decomposition meet in no more than one point. A mapping and the corresponding decomposition are finitely multiple if every element of the decomposition contains a finite number of points, and boundedly multiple if there is a natural number k such that every element contains no more than k points. A decomposition is nontrivial if there are elements containing more than one point.

The closure, interior, and boundary of a set K are denoted, respectively, by $[K]$, $\operatorname{Int} K$, $\operatorname{Fr} K$, and the empty set by Λ.

2. **Theorem 1.** *An open-and-closed finitely multiple continuous mapping of a connected manifold has bounded multiplicity.*

Theorem 2. *If θ is a completely continuous boundedly multiple decomposition of a connected manifold, then its elements of maximal multiplicity form an open everywhere dense set.*

Theorem 3. *For every open subset H with compact closure of a connected manifold there exists a positive number ε such that every nontrivial completely continuous decomposition of bounded multiplicity contains elements that meet H and have diameter greater than ε.*

3. **Notation and plan of the proof.** Let M be a connected manifold, $f : M \to X$ an open-and-closed finitely multiple continuous mapping,

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and θ the corresponding completely continuous decomposition. Denote by K_i the union of elements of multiplicity not exceeding i, and let $\Phi_i = K_i \setminus K_{i-1}$, $E_i = [\operatorname{Int} \Phi_i]$. All K_i, Φ_i, and E_i consist of whole elements of the decomposition. On each Φ_i the mapping is locally homeomorphic and strictly i-fold.

It may be assumed that M has no boundary, since the natural extension of f to the double of M reduces all the theorems to this case. Let the dimension of M be n.

First Theorems 2 and 3 are proved. The proof of Theorem 1 uses Theorem 2. Theorem 2 is proved by induction on the maximal multiplicity.

4. Let first the maximal multiplicity be equal to two. Interchanging the points belonging to one element of the decomposition, we obtain a continuous involution. One can apply the theorems of Newman and Smith ([1,3]), from which it follows that: a) the set of one-point elements of the decomposition is nowhere dense, and b) if the dimension of this set is equal

$n-1$, and it separates the manifold, then it separates it into two components, and every element consisting of two points intersects each of them.

Suppose that the maximal multiplicity of the elements of f is equal to k, and suppose Theorem 2 is true when the maximal multiplicity is less than k.

5. $K_{k-1} \setminus K_1$ is nowhere dense. Let $E_i \neq \Lambda$. Take in $\operatorname{Fr} E_i$, $1 < i < k$, an element of the greatest possible multiplicity ξ. At least one of its points x will not be a point of local homeomorphism, but it will be such a point relative to $\operatorname{Fr} E_i$. Take a connected neighborhood Ux on which there is induced a completely continuous decomposition, whose elements from $\operatorname{Fr} E_i$ meet in no more than one point each and which $\operatorname{Fr} E_i$ divides into two nonempty open sets $V_1 \subset E_i$ and V_2, $V_2 \cap E_i = \Lambda$. The decomposition induced on U is nontrivial. If x does not coincide with ξ, then the multiplicity of the decomposition induced on U and, consequently, on each V_j, is less than k, and on at least one V_j the decomposition is nontrivial. If x coincides with ξ, then the multiplicity of the decomposition induced on V_1 is equal to i, i.e. the decomposition is nontrivial and the multiplicity is less than k. In both cases, on at least one V_j there is induced a nontrivial completely continuous decomposition of multiplicity less than k. The points separating V_1 and V_2 serve in U as one-point elements of the decomposition. Therefore, if on this V_j we preserve the old decomposition, and declare all points of the other to be one-point elements, we obtain a nontrivial completely continuous decomposition of multiplicity less than k, with the one-point elements filling a nonempty open subset. Since U is connected, this contradicts the induction hypothesis. It remains to show that $\operatorname{Int} K_1 = \Lambda$.

6. The dimension of Φ_i, $1 < i < k$, is not greater than $n-1$. Suppose that for some i_0, $i < i_0 < k$, $\dim \Phi_{i_0} = n-1$. Let x be a point of an

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element ξ lying in Φ_{i_0}, and let at it Φ_{i_0} locally separate M. Since f is locally homeomorphic on Φ_{i_0}, it also locally separates M at all the other points $\xi\Phi_{i_0}$. At least one point of ξ, say x, will not be a point of local homeomorphism. As above, take a connected neighborhood Ux so that on it θ induces a completely continuous decomposition, so that the elements from Φ_{i_0} meet it in no more than one point and so that Φ_{i_0} divides it into two nonempty open sets V_1 and V_2. Preserving on one of them the old decomposition and considering the points of the other as one-point elements of the decomposition, we obtain on U a completely continuous decomposition of multiplicity less than k. By the induction hypothesis it is trivial. Therefore every element θ meets each V_j in no more than one point, and the multiplicity of the decomposition which θ induces on U is equal to 2. Hence in U the decomposition is arranged as described in item 4b). Since the image of U is not locally Euclidean, no point ξ is a point of local homeomorphism, and the same reasoning is applicable to each of them. Therefore near each point ξ the decomposition is arranged in the same way. We note that near ξ there may be either elements of multiplicity i_0, or elements of multiplicity $2i_0$. Since there must be elements of maximal multiplicity there, $2i_0 = k$. Therefore only one Φ_{i_0} can have, at some of its points, dimension $n-1$, and then k is even. It is shown that, as described in item 4, the decomposition looks near every point of Φ_{i_0} at which Φ_{i_0} has dimension $n-1$. Denote the $(n-1)$-dimensional part of Φ_{i_0} by $\Phi_{i_0}^{n-1}$. It is open in K_{k-1}.

7. Special homologies. As the coefficient domain we take Z_p, where p is the least prime divisor of k (if $\Phi^{n-1} \neq \Lambda$, $p = 2$). Homologies are considered on special coverings (the technique for constructing them is the same as in Smith ([4])). A locally finite covering of a manifold is special if: 1) it is n-dimensional*; 2) the nerves of n-dimensional simplexes

* The dimension of a covering is equal to the multiplicity decreased by one; an r-dimensional nerve of a covering is a nonempty intersection of $r+1$ elements.

the nerve lie either in Φ_k, or in $\operatorname{Int} K_1$, while the $(n-1)$-dimensional ones either intersect $K_1 \cup \Phi^{n-1}$, but not $K_{k-1} \setminus (K_1 \cup \Phi^{n-1})$, or are contained in Φ_k; 3) its elements decompose into finite systems, called marked systems, with the following properties: a) the elements of one marked system together form an open set G, consisting of whole elements of the partition, and are its components; b) every element of the partition in G intersects each of its components, and at least one of them intersects each component in a single point; 4) marked systems that intersect Φ^{n-1} and do not intersect $K_{k-1} \setminus \Phi^{n-1}$ contain $k/2$ elements; a marked system lying in Φ_k contains k elements, and if two such systems intersect one another, then two elements of the system Φ^k intersect each element of the system Φ^{n-1}.

Special coverings form a cofinal part in the set of all coverings. If a special covering is inscribed in a special one, then there exist special projections carrying

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marked systems into marked systems.

Together with the elements, the oriented simplexes of the nerve decompose into marked systems.

Define an operator σ, analogous to the corresponding Smith operator. If a chain C of dimension $n-1$ or n is considered on the nerve of a special covering, then on simplexes whose carriers do not lie in Φ_k, the chain σC takes the value zero, while on simplexes in Φ_k the value of the chain σC is equal to the sum of the values that the chain C takes on the simplexes belonging with the given one to a single marked system. For an n-dimensional simplex δ^n,

$$\sigma\partial\delta^n = \partial\sigma\delta^n.$$

This is obvious if the carrier of δ^n intersects K_1. Suppose the carrier of δ^n lies in Φ_k, and suppose first that the carrier of δ^{n-1} intersects K_1. Then δ^{n-1} will be a face of all k simplexes (or of none) from the marked system to which δ^n belongs, and mod$p$ the right-hand chain on δ^{n-1} is, like the left-hand one, zero. If the carrier of δ^{n-1} does not intersect K_1, but intersects Φ^{n-1}, then from the system containing δ^n it bounds exactly two simplexes, and $\bmod 2$ again both chains are equal to zero. If, finally, the carrier of δ^{n-1} lies in Φ_k, then either every simplex of the marked system δ^{n-1} serves as the face of exactly one simplex of the system δ^n, or these systems have no incident simplexes. Accordingly, either both parts of the equality on δ^{n-1} are equal to ± 1, or to zero. From the equality just proved it follows that on special coverings σ carries n-dimensional cycles into n-dimensional cycles. The operator σ commutes with the special projections, and its action carries over to the n-dimensional cycles of M.

8. **End of the proof of Theorem 2.** If Δ is a basic cycle of M, then the carrier of the cycle $\sigma\Delta$ lies outside K_1, and either $\sigma\Delta = 0$, or K_1 is nowhere dense. If $\sigma\Delta = 0$, then on the nerve of every special covering $\mathcal{U}$ the sum of the coefficients of the cycle $\Delta(\mathcal{U})$ on the simplexes of one marked system from Φ_k is zero. Then it is verified that the boundary of the part of $\Delta(\mathcal{U})$ lying outside K_1 is equal to zero, and Δ decomposes into the sum of two cycles $\Delta_1 + \Delta_2$, of which the carrier of Δ_2 is K_1, and the carrier of Δ_1 is $[M \setminus K_1]$. But then one of them, obviously Δ_2, must be zero, and hence Δ lies on $[M \setminus K_1]$. Since Δ is nonzero, $[M \setminus K_1] = M$ and K_1 is nowhere dense.

9. **The proof of Theorem 3** is entirely analogous to the proof of the corresponding assertion of Smith ((1), p. 457), and is therefore omitted here.

10. **Proof of Theorem 1.** Let θ be a completely discontinuous partition of finite, but unbounded, multiplicity. It is proved by a method analogous to the proof of item 5 that the boundary E_i has the following properties: 1) the set of elements of multiplicity strictly equal to i is everywhere dense on it, and 2) near each element there are elements of arbitrarily high

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multiplicity. Consider the closed set

$$\Pi = M \setminus \bigcup_i \operatorname{Int} K_i.$$

It consists of whole elements which are characterized by the fact that, near each of them, there lie elements of arbitrarily high multiplicity. From 2) it follows that: 3) $\bigcup \operatorname{Fr} E_i \subset \Pi$. Π is the sum of its intersections with K_i, and from Baire' s theorem it follows that there is an open subset U of the manifold, consisting of whole elements of the decomposition and containing elements of Π, but the elements of Π in it are bounded in multiplicity. By the property of the elements of Π, the multiplicities of the elements in U will not be bounded. Therefore, in U there are, for arbitrarily large values of i, elements of E_i, and hence also of $\operatorname{Fr} E_i$. But, by properties 1) and 3), $\Pi \cap U$ then contains elements of arbitrarily high multiplicity, which contradicts the choice of U.

Remark. From the proof of Theorem 2 it follows that, if $f : M \to M'$ is a finite-multiple open-closed mapping of a manifold without boundary onto a manifold without boundary, then the dimension of the set of points which are not points of local homeomorphism does not exceed $n-2$.

Addendum. In connection with results of Hemmingsen and Church, the following proposition is of interest; it frees one theorem of [5] from unnecessary restrictions.

Theorem. Let $f : X \to Y$ be an open-closed continuous mapping of a continuum manifold with compact inverse images of points, and let the dimension of the set B_f of points which are not points of local homeomorphism not exceed $n-2$. Then either f has bounded multiplicity, or $fB_f = fX$, and the sum of the elements of infinite multiplicity forms an everywhere dense set.

The proof uses the same set-theoretic considerations as above and, in addition to Kuratowski' s theorem, to which Hemmingsen and Church refer, also relies on Baire' s theorem.

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