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THE FIRST AND SECOND BOUNDARY-VALUE PROBLEMS FOR ELLIPTIC EQUATIONS DEGENERATING AT A FINITE NUMBER OF INTERIOR POINTS

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Equations of elliptic type that allow degeneration of order, or, in other terminology ([4,5]), equations with singular coefficients, are the subject of the works ([1,2,4,5]). In the works of L. G. Mikhailov ([4,5]), equations were investigated that degenerate on certain manifolds M_k of dimension k ($0 \le k \le n-1$; n is the number of independent variables in the equation). He studied the manifold of solutions, the first and second boundary-value problems, and solutions in the whole space. The apparatus of the investigation is a new class of special integral equations ([5]). In the case $k = 0$, he considered the equation

$$r^\alpha \Delta u + r^\alpha - 1 \sum_{i=1}^{n} b_i(x) u_{x_i}(x) + c(x)u(x) = f(x)$$

for $\alpha = 2$, $n > 2$ and for $\alpha < 2$, $n = 2$; here r is the distance from the point x to the point of degeneration. Generalized solutions of elliptic equations degenerating at isolated points were considered by V. K. Zakharov ([2]), and those degenerating on manifolds M_k by V. P. Glushko ([1]). In the case of a second-order equation, point singularities of order $\alpha < 2$ were studied. Singularities of higher order are admitted only for a very particular type of equations.

In the present work we study the generalized maximum principle, the first and second internal and external boundary-value problems, and solutions of the equation in the whole space. All these results are obtained for any equation for which the generalized maximum principle holds ([7]).

1. Internal boundary-value problems

Let a finite closed domain S with boundary Γ be given, and let the origin O be interior to S. Denote by S_0 the domain $S - \Gamma - O$. Consider in S the elliptic equation

$$Lu \equiv r^{\mu} \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j}(x) + r^{\nu} \sum_{i=1}^{n} b_i(x) u_{x_i}(x) + c(x) u(x) = f(x). \quad (1)$$

Here $x = (x_1, x_2, \dots, x_n)$ denotes a point of n-dimensional space, r is the distance from the point x to the origin, and μ and ν are constants. Suppose that the coefficients of equation (1) satisfy the condition of uniform ellipticity of the operator L in any closed domain $S - \overline{B}(O, \varepsilon)$, where $\overline{B}(O, \varepsilon)$ is the ball with center at the origin and arbitrarily small radius ε. Assume the coefficients $c(x) \leq 0$, $a_{ij}(x)$, $b_i(x)$ to be bounded in S. The functions $a_{ij}(x)$ are continuous at the point $x = 0$, and, without loss of generality, we may suppose that they have the form

$$a_{ij}(0) = \begin{cases} 1, & \text{if } i = j = 1, 2, \dots, m; \quad 0 \leq m \leq n, \\ 0, & \text{for the remaining } i, j = 1, 2, \dots, n. \end{cases} \quad (2)$$

Denote by $C^2(S, \beta)$ the class of functions $u(x)$ twice continuously differentiable in S_0 and satisfying the condition $u(x) = o(r^{-\beta})$ as $r \to 0$, where β is a positive number. We shall call the operator L β-normal if, in the ball $B(O, \delta)$ of sufficiently small radius δ, the inequality

$$Lr^{-\beta} \leqslant 0 \quad (3)$$

holds.

Let us note some special cases in which condition (3) is fulfilled.

A. Let $m > 2$ and $c(x) \leqslant 0$. Then inequality (3) will hold provided

$$\sup_{0 \leqslant r \leqslant \delta} \left(r^{\nu - \mu} \sum_{i=1}^{n} b_i(x) x_i \right) < m - 2 - \beta.$$

B. Let $m \geqslant 2$, $2 \leqslant \mu \leqslant \nu + 1$, and $\sup c(x) < 0$ in $B(O, \delta)$. Then, whatever the bounded coefficients $c(x) \leqslant 0$ and $b_i(x)$ may be, there exists $\beta > 0$ such that inequality (3) is satisfied.

Theorem 1. Let $u(x)$ be a nonconstant function, continuous in $S_0 + \Gamma$, of class $C^2(S, \beta)$, satisfying in S_0 the inequality $Lu \geqslant 0$, where the operator L is β-normal. Then, if $c(x) \equiv 0$,

$$u(x) \leqslant \max_{\Gamma} u(x)$$

in S, and equality is possible only on Γ; if $c(x) \leqslant 0$, then

$$u(x) \leqslant \max_{\Gamma}(u(x), 0)$$

in S, and equality is possible only on Γ and at the point $x = 0$.

From Theorem 1 one easily obtains analogues of Giraud' s theorem and of the uniqueness theorem for solutions of boundary-value problems in the class of functions $C^2(S, \beta)$ (7). Suppose that on Γ one of the conditions is prescribed:

$$u(x)\big|_{\Gamma} = \Psi(x); \tag{4}$$

$$\frac{\partial u}{\partial \bar{\nu}} + b(x)u = \Psi(x), \tag{5}$$

where $\bar{\nu}$ is the exterior conormal to Γ with respect to S.

Theorem 2. Let the coefficients $c(x) \leqslant 0$, $a_{ij}(x)$, $b_i(x)$, and $f(x)$ be bounded in the closed domain S and of class $A^{(1,\lambda)}$, i.e. Γ is a Lyapunov surface, and let them satisfy the conditions

$$a_{ij}(x) \in C^{(1,\lambda)}, \qquad b_i(x), c(x) \in C^{(0,\lambda)} \quad \text{in } S_0 + \Gamma,$$

the function $f(x)$ is continuous in $S_0 + \Gamma$ and belongs to the class $C^{(0,\lambda)}$ in S_0. Let the operator L be β-normal for some $\beta > 0$, and suppose there exists a sufficiently small ball $B(O, R)$ such that $f(x) \equiv 0$ in it. Then the problem (1), (4) for any continuous function $\Psi(x)$ (the problem (1), (5) for any continuous $b(x) \geqslant 0$ and $\Psi(x)$, with at least one of the functions $c(x)$ and $b(x)$ not identically zero) has, and moreover has uniquely, a solution $u(x)$ bounded in S.

Theorem 3. Let the conditions of Theorem 2 be satisfied, except for the equality $f(x) \equiv 0$, in place of which we require boundedness of $f(x)/c(x)$ in some ball $B(O, R)$. Then the assertion of Theorem 2 is valid.

If the function $f(x)/c(x)$ has the unique limit

$$(f/c)_0 = \lim_{x \to 0} f(x)/c(x)$$

and the inequality $Lr^{\beta_1} \leqslant 0$ is satisfied in $B(O, d)$ for some $\beta_1 > 0$ and $d > 0$, then the solution $u(x)$ will have the unique limit

$$(u)_0 = \lim_{x \to 0} u(x),$$

and moreover the equality

$$(u)_0 = \left(\frac{f}{c}\right)_0. \tag{6}$$

Formula (6) indicates the singular character of equation (1), since in the regular case the value $(u)_0$ depends on the boundary condition.

Remark to Theorems 2 and 3. Theorems 2 and 3 remain valid if, in the condition of β-normality (3), instead of $r^{-\beta}$ one takes an arbitrary function $w(x) > 0$ such that $w(x) \to \infty$ as $x \to 0$ and $Lw \leqslant 0$ in $B(0, d)$. In the second condition of Theorem 3, instead of r^{β_1} one may take a function $v(x) > 0$ such that $v(x) \to 0$ as $x \to 0$ and $Lv \leqslant 0$ in $B(0, d)$.

2. External boundary-value problems. Denote by S^- the domain which is the complement of the bounded closed domain S to the whole space E_n. Consider in S^- the elliptic equation

$$L'u \equiv r^{\mu'} \sum_{i,j=1}^{n} a'_{ij}(x) u_{x_i x_j}(x) + r^{\nu'} \sum_{i=1}^{n} b'_i(x) u_{x_i}(x) + c'(x)u(x) = f'(x), \tag{1'}$$

where μ', ν' are constants. Suppose that the operator L' is uniformly elliptic in any closed domain $B(0, A) - S$, where $A > 0$ is sufficiently large. Let the coefficients $c'(x) \leqslant 0$, $a'_{ij}(x)$, $b'_i(x)$, and $f'(x)$ be bounded in S^-, and let the functions $a'_{ij}(x)$ have, respectively, unique limits which are defined by the equalities

$$(a'_{ij})_\infty = \lim_{x\to\infty} a'_{ij}(x) = \begin{cases} 1 & \text{for } i = j = 1, 2, \dots, m'; \quad 0 \leqslant m' \leqslant n, \\ 0 & \text{for the remaining } i, j = 1, 2, \dots, n. \end{cases} \tag{2'}$$

Denote by $C^2(S^-, \beta^-)$ the class of functions $u(x)$, twice continuously differentiable in S^-, satisfying the condition $u(x) = o(r^{\beta'})$ as $r \to \infty$, where β' is a positive number. We shall call the operator L' β'-normal if, outside some ball $B(0, B)$ of sufficiently large radius B, the inequality

$$L' r^{\beta'} \leqslant 0 \tag{3'}$$

is satisfied.

For β'-normal operators, theorems analogous to Theorems 1–3 hold. We state two of them.

Theorem 2′. Suppose the coefficients $c'(x) \leqslant 0$, $a'_{ij}(x)$, $b'_i(x)$, $f'(x)$ are bounded in the closed domain S^-, $S \in A^{(1,\lambda)}$, and satisfy the conditions: $a'_{ij}(x) \in C^{1,\lambda}$; $b'_i(x)$, $c'(x) \in C^{(0,\lambda)}$ in $S^- + \Gamma$, $f'(x)$ is continuous in $S^- + \Gamma$, and u belongs to the class $C^{(0,\lambda)}$ in S^-. Suppose the operator L' is β'-normal for some $\beta' > 0$ and that there exists a ball $B(0, R_1)$ so large that $f'(x) \equiv 0$ outside it. Then problem (1′), (4) for any continuous function $\Psi(x)$ (problem (1′), (5) for any continuous $b(x) \leqslant 0$ and $\Psi(x)$, with, among the functions $c'(x)$ and $b(x)$, at least one not identically equal to zero) has, and moreover has uniquely, a solution $u(x)$ bounded in S^-.

Theorem 3′. Suppose the conditions of Theorem 2′ are fulfilled, except for the equality $f'(x) \equiv 0$, in place of which we require the boundedness of $f'(x)/c'(x)$ outside some ball $B(0, R_1)$. Then the assertion of Theorem 2′ is valid.

If the function $f'(x)/c'(x)$ has a unique limit

$$(f'/c')_\infty = \lim_{x\to\infty} f'(x)/c'(x)$$

and the inequality $L' r^{-\beta_1} = 0$ is satisfied outside $B(0, d)$ for some $\beta_1 > 0$ and $d > 0$, then the solution $u(x)$ will have a unique limit

$$(u)_\infty = \lim_{x\to\infty} u(x),$$

and the equality

$$(u)_\infty = \left(\frac{f'}{c'}\right)_\infty \tag{7'}$$

holds.

3. Solution of the equation in the whole space

Consider the elliptic equation

$$\mathcal{L}u \equiv \sum_{i,j=1}^{n} A_{ij}(x) u_{x_i x_j}(x) + \sum_{i=1}^{n} B_i(x) u_{x_i}(x) + C(x)u(x) = F(x) \tag{8}$$

in the whole space E_n. Denote by $K(a, b)$ the domain $a \le r(x) \le b$. Suppose that the operator $\mathcal{L}$ is uniformly elliptic in every domain $K(\varepsilon, \varepsilon^{-1})$, where ε is an arbitrarily small positive number. Let equation (8) reduce in some ball $B(O, \delta)$ to equation (1), and outside the ball $B(O, \delta^{-1})$ to (1′). Consequently, we shall consider an equation with two singular points O, ∞ (cf. (5)).

Theorem 4. Let the coefficients of the operator $\mathcal{L}$ in any domain $K(\varepsilon, \varepsilon^{-1})$ satisfy the conditions:

$$A_{ij}(x) \in C^{(1,\lambda)}; \qquad B_i(x),\ C(x) \le 0; \qquad F(x) \in C^{(0,\lambda)}.$$

Then the following results hold:

1) If there exists such a small R that $F(x) \equiv 0$ in $B(O,R)$, the conditions of Theorem 2 are satisfied, and all the conditions of Theorem 3′ are satisfied outside $B(O,R^{-1})$, then there exists, and moreover uniquely, a bounded in E_n solution $u(x)$ of equation (8), regular for $0 < r < \infty$, for which equality (7) holds.

2) If the first part of the conditions of Theorem 3 is satisfied in $B(O,R)$ and all the conditions of Theorem 3′ outside $B(O,R^{-1})$, then assertion 1) holds. If the second part of Theorem 3 is satisfied, then the solution $u(x)$ satisfies equality (6).

We note that in Theorem 4 the points O and ∞ may be interchanged, and we obtain a new result analogous to Theorem 4.

The proof of Theorems 2–4 is carried out according to the following scheme. By a method analogous to that used by M. V. Keldysh (3), the existence of a solution of the first boundary-value problem for the ball $B(O,R)$ or for the exterior of the ball $B(O,R^{-1})$ is proved. Uniqueness of the solution follows from the generalized maximum principle. For domains of general form the boundary-value problems are solved by a modified method of alternation (8), based on the results of (7).

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Note: Figure translations are in progress. See original paper for figures.