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Abstract

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SOME PROPERTIES OF MANIFOLDS OF DIMENSION $4k + 2$

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The present paper is adjacent to the author' s works ([8]) on the diffeomorphism of simply connected manifolds and to Kervaire' s work on the existence of nonsmoothable manifolds of dimension 10 ([4]). The central part of the paper, devoted to properties of 10-dimensional manifolds, also essentially uses ideas from works of Milnor and the author ([5], [9]) concerning generalized rings of internal homologies. Recall (see ([8])) that in studying the homotopy group $\pi_{N+n}(T_N)$ of the Thom space T_N of the normal bundle to a manifold M^n $(n = 4k + 2)$, we singled out a subset $A \subset \pi_{N+n}(T_N)$, consisting of those elements $a \in A$ for which $H(a) = \varphi[M^n]$, where $\varphi : H_k(M^n) \to H_{k+N}(T_N)$ is the Thom isomorphism and $H : \pi_j(x) \to H_j(x)$ is the Hurewicz homomorphism. It was proved that a regular representative $f_a : S^{N+n} \to T_N$ of an element $a \in A$ can be chosen so that the manifold $f_a^{-1}(M^n) = M_a^n$ has the following properties:

1. $f_{a*} : H_i(M_a^n) \to H_i(M^n)$ is an isomorphism for $i \neq 2k + 1$.
2. $\operatorname{Ker} f_{a*} = Z + Z \subset H_{2k+1}(M_a^n)$.
3. The Hurewicz homomorphism $H : [\operatorname{Ker} f_{a*} \to \operatorname{Ker} f_{a*} \subset H_{2k+1}(M_a^n)]$ is an isomorphism.
4. If a cycle $x \in \operatorname{Ker} f_{a*}$ is realized by an embedded sphere $S^{2k+1} \subset M_a^n$ and $n \neq 6, 14$, then the normal bundle $\nu(S^{2k+1}, M_a^n)$ of the sphere S^{2k+1} in the manifold M_a^n depends only on the element x, belongs to the group Z_2, and determines a mapping $\varphi : \operatorname{Ker} f_{a*} \to Z_2$ such that

$$\varphi(x + y) = \varphi(x) + X(y) + x \cdot y \mod 2.$$

If x, y is a basis of the group $\operatorname{Ker} f_{a*}$, then we set $\varphi(a) = \varphi(x)\varphi(y) \in Z_2$.

Theorem 1. *The invariant $\varphi(a)$ does not depend on the choice of the representative f_a having properties 1-4.*

If $n = 6, 14$, then the definition of the invariant must be changed: namely, in item 4 one should speak not of the normal bundle of the sphere S^{2k+1} in the manifold M_a^n, but of the "framing" of the sphere. The exact definition of the invariant is given in ([10]), and it is denoted by $\psi(a) \in Z_2$. All algebraic properties of the invariant $\varphi(a)$ carry over to the invariant $\psi(a)$ for $n = 6, 14$.

Theorem $1'$**.** *The invariant $\psi(a)$ is uniquely and correctly defined.* The proof of Theorem $1'$ is identical to the proof of Theorem 1.

By $\widetilde{A} \subset A$ we shall denote the subset of the set $A \subset \pi_{N+n}(T_N)$ consisting of those elements $a \in \widetilde{A}$ for which $\varphi(a) = 0$ (for $n \neq 6, 14$) and $\psi(a) = 0$ ($n = 6, 14$). From the definition of the invariants φ and ψ and from Theorem 1, Corollaries 1 and 2 easily follow.

Corollary 1. *For $n = 6, 14$, the set $\widetilde{A}$ contains exactly half of the elements of the set A, independently of the manifold M^n.*

Corollary 2. *If $M^n = M_1^n \# M_2^n$, then the set $\widetilde{A}$ coincides with the set A for the manifold M^n if and only if $\widetilde{A}$ coincides with A for each of the manifolds M_1^n and M_2^n.*

We now study the distribution of the invariant φ on the set A. Suppose, for simplicity, that the manifold M^n is such that $H^{2k+1}(M^n, Z) \otimes Z_2 = 0$ and $n = 4k+2,\ k \neq 0, 1, 3$.

Let us note that $\pi_{N+n}(\widetilde{T}_N) = Z + \tilde{\pi}$, where the group $\tilde{\pi}$ is finite. The set $\widetilde{A}$ is the collection of all elements of the form $1 + \gamma$, where $1 \in Z$ and $\gamma \in \tilde{\pi}$. It is assumed that the decomposition of the group $\pi_{N+n}(T_N)$ into a direct sum has been chosen and fixed.

Theorem 2. The formula

$$\varphi(1 + \gamma + \delta) = \varphi(1 + \gamma) + \varphi(1 + \delta) + \varphi(1 + 0), \qquad \text{where } \gamma, \delta \in \tilde{\pi},$$

holds.

The proof of Theorem 2 is not difficult and follows from the representation

$$1 + \gamma + \delta = (1 + \gamma) + (1 + \delta) - (1 + 0),$$

on the basis of which the element $1 + \gamma + \delta$ can be realized by a "good" representative $f : S^{N+n} \to T_N$, and one can trace the surgeries of the full inverse image $f^{-1}(M^n)$.

In our case the direct decomposition $\pi_{N+n}(T_N) = Z + \tilde{\pi}$ can be chosen so that $\varphi(1 + 0) = 0$; therefore, putting $\varphi(\gamma) = \varphi(1 + \gamma)$, we obtain a homomorphism $\varphi : \tilde{\pi} \to Z_2$. Hence follows

Corollary 3. If $H^{2k+1}(M^n, Z) \otimes Z_2 = 0$, then the set $\widetilde{A}$ contains either half of the set A, or coincides with the set A.

We have carried out a study of the relation between the sets $\widetilde{A}$ and A, sufficiently complete for dimension $n = 4k + 2$ with $k = 1, 3$, and giving some information about higher dimensions. The first nontrivial case is $k = 2$ ($n = 10$), on which we shall dwell further. The nontriviality of this case is due to the fact that for the sphere the sets $\widetilde{A}$ and A coincide, and it is quite unclear what the situation will be for other manifolds. Our goal is a generalization of the invariant $\varphi(M^{10}) \in Z_2$, defined by Kervaire ([4]) for 4-connected 10-dimensional manifolds, and the application of the Kervaire invariant to the solution of certain problems.

Since the cohomology operation

$$\mathrm{Sq}^2\,\mathrm{Sq}^4 : H^5(X, Z) \to H^{11}(X, Z_2)$$

is identically zero by virtue of the relation

$$\mathrm{Sq}^2\,\mathrm{Sq}^4 = \mathrm{Sq}^6 + \mathrm{Sq}^5\,\mathrm{Sq}^1,$$

there is defined a "secondary" cohomology operation

$$\Phi : \mathrm{Ker}\,\mathrm{Sq}^4 \to \mathrm{Coker}\,\mathrm{Sq}^2, \qquad \text{where } \mathrm{Ker}\,\mathrm{Sq}^4 \subset H^5(X, Z).$$

Lemma 1. The operation Φ has the following property:

$$\Phi(x + y) = \Phi(x) + \Phi(y) + xy.$$

The proof of Lemma 1 is very simple.

Lemma 2. If $\pi_1(M^{10}) = 0$ and $w_2(M^{10}) = 0$ for a topological manifold M^{10}, then the operation

$$\Phi : H^5(M^{10}, Z) \to H^{10}(M^{10}, Z_2) = Z_2$$

is always defined and unambiguous.

Lemma 3. Under the conditions of Lemma 2, the operation Φ defines a unique homomorphism

$$\Phi : \mathrm{Tor}\, H^5(M^{10}, Z) \to H^{10}(M^{10}, Z_2).$$

Lemma 3 follows easily from Lemmas 1 and 2.

In what follows we shall study only manifolds having the following properties:

1. $\pi_1(M^{10}) = 0$.
2. $w_2(M^{10}) = 0$.
3. The homomorphism $\Phi : \mathrm{Tor}\, H^5(M^{10}, Z) \to H^{10}(M^{10}, Z_2)$ is trivial.

For topological manifolds possessing properties 1-3, we define the "generalized Kervaire invariant" as follows: a) choose in the group $H^5(M^{10}, Z)/\mathrm{Torsion}$ a basis $x_1, \dots, x_{2l}$ such that $x_{2i-1}x_{2i} \neq 0$ for $1 \leq i \leq l$ and $x_{kx}s = 0$ otherwise; b) by virtue of condition 3, the operation Φ is uniquely defined on the group $H^5(M^{10}, Z)/\mathrm{Torsion}$ and takes its values in Z_2; the sum

$$\Phi(M^{10}) = \sum_{i=1}^{l} \Phi(x_{2i-1})\Phi(x_{2i})$$

does not depend on the basis and is called the "generalized Kervaire invariant" ; c) the invariant $\Phi(M^{10})$ is a homotopy invariant of the manifold.

The following important result holds.

Lemma 4. *If the manifold M^{10} is smooth and is the boundary of a smooth oriented manifold W^{11} such that $w_2(W^{11}) = 0$, then $\Phi(M^{10}) = 0$.*

By analogy with the papers ([5],[9]) we shall consider the ring of spinor internal homologies ("cobordisms")

$$V_{\text{Spin}} = \sum V_{\text{Spin}}^i, \qquad V_{\text{Spin}}^i = \pi_{N+n}(M\text{Spin}\, N),$$

where $M\text{Spin}\, N$ is the Thom complex of the spinor group. As is known, the tangent (or stable normal) bundle of a manifold reduces to the group Spin if $w_1(M^n) = w_2(M^n) = 0$. Thus, from Lemma 4 the following follows.

Lemma 5. *The Kervaire invariant defines a single-valued homomorphism*

$$\Phi : \widetilde{V}_{\text{Spin}}^{10} \to Z_2, \qquad \text{where } \widetilde{V}_{\text{Spin}}^{10} \subset V_{\text{Spin}}^{10}.$$

Proof. The additivity of the invariant Φ is obvious. If a manifold defines the zero element of the group V_{Spin}^{10}, then there is stretched over it a film W^{11} such that $w_2(W^{11}) = 0$, and therefore $\Phi = 0$. However, by virtue of restriction 3, the invariant Φ, perhaps, is not defined for all elements of the group V_{Spin}^{10}. The lemma is proved.

We give, without proof, a number of results on the ring

$$V_{\text{Spin}} = \sum V_{\text{Spin}}^i.$$

I. The groups V_{Spin}^i for $i \leq 10$ are given by the following table:

i	$=$	0	1	2	3	4	5	6	7	8	9	10
V_{Spin}^i	$=$	Z	Z_2	Z_2	0	Z	0	0	0	$Z+Z$	Z_2+Z_2	$Z_2+Z_2+Z_2$

II. The generators of the groups V_{Spin}^k for $k \leq 10$ may be chosen as follows:

$$1 \in V_{\text{Spin}}^0, \qquad x_1 \in V_{\text{Spin}}^1, \qquad x_1^2 \in V_{\text{Spin}}^2, \qquad x_4 \in V_{\text{Spin}}^4,$$

$$x_8 \in V_{\text{Spin}}^8, \qquad y_8 \in V_{\text{Spin}}^8, \qquad 4y_8 = x_4^2, \qquad x_1 x_8 \in V_{\text{Spin}}^9,$$

$$x_1 y_8 \in V_{\text{Spin}}^9, \qquad x_1^2 x_8 \in V_{\text{Spin}}^{10}, \qquad x_1^2 y_8 \in V_{\text{Spin}}^{10}, \qquad Z_{10} \in V_{\text{Spin}}^{10}.$$

III. The element $x_1 \in V_{\text{Spin}}^1$ is represented by the circle $S^1 \subset R^{N+1}$ with nontrivial framing.

IV. The group V^8_{Spin} is generated by the following manifolds: a) the quaternionic projective plane $P^2(Q)$; b) the 8-dimensional 3-connected almost parallelizable Milnor manifold M^8_0 with index $I(M^8_0) = 8 \cdot 28$.

V. The generator $Z_{10} \in V^{10}_{\text{Spin}}$ is represented by a manifold M^{10} such that

$$w_4 w_6(M^{10}) \neq 0.$$

The subgroup

$$V^1_{\text{Spin}} V^1_{\text{Spin}} V^8_{\text{Spin}} \subset V^{10}_{\text{Spin}}$$

is singled out by the condition $w_4 w_6 = 0$. The results given in items I–V may be obtained by analogy with the papers of Milnor ([5]) and the author ([9]) on rings of generalized internal homologies by means of Adams' spectral method ([1]) and the A-genus ([2]).

Lemma 6. *The homomorphism*

$$\Phi : \widetilde{V}^{10}_{\text{Spin}} \to Z_2$$

is trivial on the subgroup

$$V^1_{\text{Spin}} V^1_{\text{Spin}} V^8_{\text{Spin}} \subset V^{10}_{\text{Spin}}.$$

The proof of Lemma 6 is nontrivial and substantially uses item IV on the geometric generators of the group

$$V^8_{\text{Spin}} = Z + Z.$$

The most difficult part is the analysis of the element represented by the manifold $P^2(Q) \times S^1 \times S^1$. In essence, one must explicitly trace the Morse surgeries in the manifolds

$$M^8_0 \times S^1 \times S^1 \quad \text{and} \quad P^2(Q) \times S^1 \times S^1$$

over one-dimensional cycles.

From the lemmas it easily follows:

Theorem 3. *The invariant $\Phi(M^{10})$ is a single-valued function of the product $w_4 w_6(M^{10})$, and $\Phi(M^{10}) = 0$ if $w_4 w_6(M^{10}) = 0$, for a smooth manifold M^{10}.*

Thus,

$$\Phi = \Phi(w_4 w_6)$$

for smooth manifolds.

Remark. The author believes that $\Phi(w_4, w_6) \equiv 0$ for smooth manifolds; for this it is enough to construct such a smooth manifold M^{10} that

$$w_4 w_6(M^{10}) \neq 0 \quad \text{and} \quad \Phi(M^{10}) = 0.$$

Theorem 4. *If, for a smooth manifold M^{10}, the invariant $\Phi(M^{10})$ is defined, then for it $\tilde{A} = A$, i.e. the invariant $\varphi(\alpha) \equiv 0$ for all $\alpha \in A$.*

Proof. Let α be such that $\varphi(\alpha) = 1$. Take a representative $M^{10}_\alpha = f^{-1}_\alpha(M^{10})$, possessing properties 1-4 indicated at the beginning of the paper. Obviously, $\varphi(M^{10}_\alpha) = \varphi(M^{10}) + 1$ and $w_4 w_6(M^{10}_\alpha) = w_4 w_6(M^{10})$; since M^{10}_α is smooth, we arrive at a contradiction with Theorem 3.

Let M^{10} be a topological manifold (or, more generally, a polyhedron possessing Poincaré duality). Following the schemes of the papers of the author ([8]) and Browder ([3]), one can prove the following assertion:

Theorem 5. *If, for a polyhedron M^{10}, the invariant Φ is defined, then the following two conditions are necessary and sufficient for M^{10} to have the homotopy type of a smooth manifold: a) $\Phi(M^{10}) = \Phi(w_4 w_6)$; b) there exists an SO_N-bundle ν over M^{10} such that the cycle $\varphi[M^{10}] \in H_{N+10}(T_N)$ is spherical, where T_N is the Thom complex of the bundle ν (ν will be the normal bundle of the smooth manifold sought).*

We note that Kervaire in ([4]) constructed a 4-connected manifold satisfying condition b) and not satisfying condition a). Thus, both conditions are essential. Moreover, for $5 \leq n \leq 17$ dimension $n = 10$ alone presented difficulty (the cases $n = 6, 14$ are simple). For $n = 4k + 2$, where $k \geq 4$, new difficulties arise. The author believes that for even k a generalization of the invariant Φ is possible on the basis of the relation

$$Sq^2 Sq^{2k} = Sq^{2k+2} + Sq^{2k+1} Sq^1$$

of the Steenrod algebra and the study of the ring V_{spin}.

We indicate some results on spin cobordisms.

Lemma 7. *If $\pi_1(M^n) = 0$ and $w_2(M^n) = 0$, then the stable normal (tangent) SO_N-bundle reduces to the group $\operatorname{Spin} N$, and uniquely. For $n \geq 3$ every element of the group V^n_{spin} is represented by a simply connected manifold.*

Consider the natural homomorphism "forgetting the framing"

$$\rho : G_i \to V^i_{\text{spin}},$$

where $G_i = \pi_{N+i}(S^N)$. The following very substantial fact holds.

Lemma 8. *For $3 \leq i \leq 8$ the image of the homomorphism $\rho : G_i \to V^i_{\text{spin}}$ is trivial. For $i = 9, 10$ the image of the homomorphism ρ is isomorphic to Z_2.*

For the proof one must consider the Milnor manifold M^8_0 indicated above and perform surgeries in the manifold $M^8_0 \times S^1$ for $i = 9$ and $M^8_0 \times S^1 \times S^1$ for $i = 10$, dragging along the nontrivial "spin framings." As a result, the corresponding elements of the groups V^9_{spin}, V^{10}_{spin} are realized by homotopy spheres.

Theorem 6. *There exist smooth manifolds of the homotopy type of spheres of dimensions 9 and 10 which are not boundaries of any smooth manifolds with trivial Stiefel class* $w_2 = 0$.

Corollary. *For* $n = 9, 10$*, membership of a smooth simply connected manifold in the class of spinor inner homologies is not a combinatorial invariant (in contrast to the usual inner homologies, corresponding to the groups* O *and* SO*) and is not determined by the homotopy type and the tangent bundle.*

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Note: Figure translations are in progress. See original paper for figures.