PHYSICS

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1963

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Abstract

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PHYSICS

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INVESTIGATION OF THE ANALYTIC PROPERTIES OF LADDER DIAGRAMS BY THE METHOD OF COMPLEX ORBITAL MOMENTA

(Presented by Academician N. N. Bogolyubov, 17 IX 1962)

Regge' s investigations on the application of complex orbital momenta (^{1}) provide a convenient method for studying the analytic properties of ladder diagrams, for which, as is known, the Mandelstam representation (^{2}) is valid. In the most general case, a ladder of order n is an analytic function of two invariants $s = (p_1 + p_2)^2$, $t = (p_1 + p_3)^2$, and, in addition, depends on the values of $4 + n/2 + 2(n/2 - 1)$ different masses, namely 4 external (m_i), $n/2$ vertical (μ_i), and $2(n/2 - 1)$ horizontal (λ_i, ν_i) masses. A cross on a line denotes the replacement of the full propagation function of the corresponding internal particle by a δ-function. It will be shown below that such a diagram has a spectral representation in t with spectral function $\rho^{(n)}(s,t)$, whose value is determined by the same diagram (see Fig. 1), but in which all propagation functions of the internal lines are replaced by δ-functions.

Fig. 1

Fig. 1

1. First we shall investigate the case in which the masses of all particles under consideration are identical and equal to m. We obtain the expansion of this diagram in Legendre polynomials. To do this, note that the expansion of the pole term in the center-of-mass system of the first channel, by Heine' s theorem, has the form

$$\frac{g^2}{m^2 - t} = 8\pi \frac{\sqrt{s}}{q} \sum_l (2l + 1)\, \gamma Q_l(z_1) P_l(z), \tag{1}$$

where

$$q = \frac{1}{2}\sqrt{s - 4m^2}, \qquad z = 1 + \frac{2t}{s - 4m^2}, \qquad z_1 = 1 + \frac{2m^2}{s - 4m^2},$$

$$\gamma = \frac{g^2}{8\pi} \frac{1}{\sqrt{s}\sqrt{s-4m^2}}. \tag{1a}$$

Here $P_l(z)$ and $Q_l(z)$ are Legendre functions of the first and second kind, respectively. By multiplying two pole terms and integrating in the center-of-mass system of the intermediate particles (3), one can obtain the expansion of the absorptive part of the box diagram in Legendre polynomials in the case of equal masses:

$$A^{(4)}(s,t) = 8\pi \frac{\sqrt{s}}{q} \sum_l (2l+1)\, \gamma^2 Q_l^2(z_1) P_l(z). \tag{2}$$

Next, multiplying (1) by (2) and integrating in the center-of-mass system of the intermediate particles, and repeating this procedure $n/2$ times, we find the following expansion in Legendre polynomials for a diagram of order n in the case of equal masses:

$$A^{(n)}(s,t) = 8\pi \frac{\sqrt{s}}{q} \sum_l (2l+1)\, \gamma^{n/2} Q_l^{n/2}(z_1) P_l(z). \tag{3}$$

It is easy to see that the solution of the Sokolov-Gaitler equation (4–6) is readily obtained from (3) by summing the series

$$\sum_{n=2,4,\ldots} A^{(n)} i^{\,n/2-1},$$

which is a geometric progression:

$$A(s,t) = 8\pi \frac{\sqrt{s}}{q} \sum_l (2l+1) \frac{\gamma Q_l(z_1)}{1 - i\gamma Q_l(z_1)} P_l(z). \tag{4}$$

Let us now proceed to the calculation of the diagram shown in Fig. 1, in the case when all internal lines are taken on the mass shell. The assumption that the masses of all particles under consideration are equal only slightly simplifies the final results. Therefore we shall give the treatment at once in the general case.

When all the masses entering into consideration are different, but satisfy the conditions for the existence of the Mandelstam normal representation, instead of (3) we shall have the following expansion for the diagram in Fig. 1:

$$A^{(n)}(s,t) = \frac{8\pi\sqrt{s}}{q_{\mathrm{H}} q_1 \ldots q_{n/2-1} q_{\mathrm{K}}} \left(\frac{g^2}{16\pi\sqrt{s}}\right)^{n/2} \sum_l (2l+1) \prod_{j=1}^{n/2} Q_l(z_j) P_l(z). \tag{5}$$

Here $q_{\mathrm{H}}, q_1, \dots, q_{n/2-1}, q_{\mathrm{K}}$ denote the momenta in the c.m.s. of the initial, intermediate, and final particles, respectively; their relation to s is given by the formula

$$q_{\alpha\beta} = \frac{\sqrt{s-(m_\alpha+m_\beta)^2}\sqrt{s-(m_\alpha-m_\beta)^2}}{2\sqrt{s}}. \tag{6}$$

The cosine of the scattering angle z is related to t by the formula

$$z = \frac{q_{\mathrm{H}}^2 + q_{\mathrm{K}}^2 + t - \left(\sqrt{m_1^2+q_{\mathrm{H}}^2} - \sqrt{m_3^2+q_{\mathrm{K}}^2}\right)^2}{2q_{\mathrm{H}}q_{\mathrm{K}}}. \tag{7}$$

By z_j we denote the cosine of the complex angle associated with the mass of the corresponding j-th vertical particle. The expression for z_j is obtained from (7) by the substitution $t \to \mu_j^2$, $q_{\mathrm{H}} \to q_{j-1}$, $q_{\mathrm{K}} \to q_j$, $m_1 \to \nu_{j-1}$, $m_3 \to \nu_j$ (in doing this, when calculating z_1 and $z_{n/2}$ it is necessary to take $q_0 = q_{\mathrm{H}}$, $\nu_0 = m_1$ and $q_{n/2} = q_{\mathrm{K}}$, $\nu_{n/2} = m_3$). By means of the Watson-Sommerfeld transformation one can analytically continue this equality into the spectral region st, where, after calculating the discontinuity in t, we find the following expression for the diagram shown in Fig. 1, in which the propagators of all internal lines have been replaced by δ-functions:

$$\rho^{(n)}(s,t) = \frac{8\pi\sqrt{s}}{q_{\mathrm{H}}q_1\cdots q_{n/2-1}q_{\mathrm{K}}}\left(\frac{g^2}{16\pi\sqrt{s}}\right)^{n/2}\frac{1}{2i}\int_{-1/2+i\infty}^{-1/2-i\infty}(2l+1)\,dl\prod_{j=1}^{n/2}Q_l(z_j)P_l(z). \tag{8}$$

The integral appearing here cannot be evaluated in closed form for $n \geq 8$ (see below). However, one can indicate that the region in which it is different from zero has the form

$$z \geq \operatorname{ch}(\varphi_1+\varphi_2+\cdots+\varphi_{n/2}), \qquad \varphi_j = \operatorname{ar\,ch} z_j. \tag{8a}$$

This can be verified by generalizing by induction the arguments by means of which, in item 2, the equation of the Karplus curve will be obtained for the sixth-order diagram. In the case of equal masses, equation (8a) is simplified and takes the form

$$z \geq z_1^{n/2} + C_{n/2}^2 z_1^{n/2-2}(z_1^2-1) + C_{n/2}^4 z_1^{n/2-4}(z_1^2-1)^2 + \cdots, \tag{8b}$$

where z and z_j are given by formulas (1a), and $C_n^m = n!/m!(n-m)!$ are binomial coefficients. It is easy to show that, for example, in the case of $\pi\pi$-scattering the boundary curve (8b) has the asymptotes $s = 4m^2$, $t = n^2m^2$.

Expression (8) is in fact an expansion in the Mehler-Fock integral in conical Legendre functions and is real. Thus we have shown that the diagram in Fig. 1 has a spectral representation in the transferred momentum of the form

$$A^{(n)}(s,t) = \frac{1}{\pi} \int \frac{\rho^{(n)}(s,t')}{t'-t}\, dt' \tag{9}$$

with a real spectral function, different from zero in the definite region (8a).

2. For $n = 2$ (the pole term), calculation of the integral in (8) leads to a δ-shaped one-dimensional spectral function. For $n = 4$ (the box)

the integral can be easily evaluated, and it can be shown that in this way one can obtain the Mandelstam spectral function of the box diagram. To evaluate the integral (8) for $n = 6$, we proceed as follows. We use formula (21) and analytically continue the left-hand side of equality (21) by means of the Watson-Sommerfeld transformation. Calculating the discontinuities in z on both sides, we obtain:

$$\frac{1}{2i} \int\limits_{-\frac{1}{2}+i\infty}^{-\frac{1}{2}-i\infty} (2l+1)\, dl\, Q_l(z_1) Q_l(z_2) Q_l(z_3) P_l(z) = 2\pi \frac{\vartheta(z-z_0)}{\sqrt{\Delta^+(z_1 z_2 z_3 z)}} K\left(\sqrt{\frac{\Delta^-(z_1 z_2 z_3 z)}{\Delta^+(z_1 z_2 z_3 z)}}\right), \tag{10}$$

where $K(k)$ is the complete elliptic integral of the first kind, and the values z_0 and $\Delta^\pm$ are defined below. Thus, the sixth-order diagram in the Heitler approximation has a spectral representation of the form (9) with the spectral function $\rho^{(6)}(s,t)$, expressed by means of (10) through a complete elliptic integral and nonzero in the region determined by the θ-function appearing here:

$$A^{(6)}(s,z) = \frac{1}{\pi} \int\limits_{z_0}^{\infty} \frac{d\xi}{\xi - z}\, \rho^{(6)}(s,\xi), \tag{11}$$

where

$$\rho^{(6)}(s,z) = \frac{8\pi\sqrt{s}}{q_H q_1 q_2 q_K} \left(\frac{g^2}{16\pi\sqrt{s}}\right)^3 \frac{2\pi\theta(z-z_0)}{\sqrt{\Delta^+(z_1 z_2 z_3 z)}} K\left(\sqrt{\frac{\Delta^-(z_1 z_2 z_3 z)}{\Delta^+(z_1 z_2 z_3 z)}}\right). \tag{12}$$

We note that the θ-function appearing here determines on the st-plane a third-order Karplus curve, beyond which the spectral function is nonzero. For example, in the case of $\pi\pi$-scattering, the equation of the boundary curve beyond which $\rho^{(6)}(s,t)$ is nonzero has the form

$$t = 36m^2 + \frac{384m^4}{s - 4m^2} + \frac{1024m^6}{(s - 4m^2)^2}$$

(asymptotes $s = 4m^2,\ t = 36m^2$). In the same way, from the general equation $z - z_0 = 0$, the equations of the Karplus curves for other real scattering processes can be obtained, if one takes into account the kinematic relations between the cosines of the angles and the invariant variables s and t.

3. As is known, the sum containing the product of two Legendre functions of the second kind and a Legendre polynomial is, up to a factor, the absorptive part of the box diagram, and its spectral representation in z has the form

$$\sum_l (2l+1)\, Q_l(z_1)\, Q_l(z_2)\, P_l(z) = \int\limits_{\xi^+}^{\infty} \frac{1}{\xi - z} \frac{d\xi}{\sqrt{k(z_1 z_2 \xi)}}, \tag{13}$$

where

$$k(x, y, z) = x^2 + y^2 + z^2 - 2xyz - 1, \tag{14}$$

$$\xi^+ = z_1 z_2 + \sqrt{z_1^2 - 1}\sqrt{z_2^2 - 1}. \tag{15}$$

Expanding under the integral sign $1/(\xi - z)$ in a Heine series and changing the order of summation over l and integration over $d\xi$, we obtain, by virtue of the orthogonality of the Legendre polynomials, the following integral representation, expressing the product of two Legendre functions of the second kind through the integral of one function multiplied by the spectral function of the box diagram:

$$Q_l(z_1) Q_l(z_2) = \int\limits_{\xi^+}^{\infty} Q_l(\xi) \frac{d\xi}{\sqrt{k(z_1 z_2 \xi)}}, \tag{16}$$

valid for arbitrary complex z_i and integers $l \geq 0$. This property of the spectral function of the box diagram "splits" one function Q_l into two for the first time as was noted by Froissart (^{7}) (see also (^{8})). Multiplying both sides of (16) by $Q_l(z_3)$ and applying formula (16) once again, we obtain

$$Q_l(z_1)\, Q_l(z_2)\, Q_l(z_3) = \int_{\zeta^+}^{\infty} \frac{d\xi}{\sqrt{k(z_1 z_2 \xi)}} \int_{\eta^+}^{\infty} Q_l(\eta)\, \frac{d\eta}{\sqrt{k(\xi, \eta, z_3)}}.$$

Here

$$\eta^{+} = \zeta z_3 + \sqrt{\zeta^2 - 1}\sqrt{z_3^2 - 1}. \tag{17}$$

Changing the order of integration with respect to $d\xi$ and $d\eta$ in the resulting double integral, we obtain

$$Q_l(z_1)\,Q_l(z_2)\,Q_l(z_3) = \int_{\zeta^{+}_{z_3} + \sqrt{\zeta^2-1}\sqrt{z_3^2-1}}^{\infty} Q_l(\eta)\,d\eta \int_{\zeta^{+}}^{z_3\eta - \sqrt{z_3^2-1}\sqrt{\eta^2-1}} \frac{d\xi}{\sqrt{k(z_1 z_2 \xi)}\sqrt{k(\xi \eta z_3)}},$$

and, evaluating the elliptic integral appearing here, we shall finally have the following integral representation for the product of three Legendre functions of the second kind:

$$Q_l(z_1)\,Q_l(z_2)\,Q_l(z_3) = \int_{z_0}^{\infty} Q_l(\eta)\,\frac{2\,d\eta}{\sqrt{\Delta^{+}(z_1 z_2 z_3 \eta)}}\,K\left(\sqrt{\frac{\Delta^{-}(z_1 z_2 z_3 \eta)}{\Delta^{+}(z_1 z_2 z_3 \eta)}}\right), \tag{18}$$

where

$$\Delta^{\pm}(z_1 z_2 z_3 z) = z_1^2 + z_2^2 + z_3^2 + z^2 - 2 z_1 z_2 z_3 z \pm$$

$$\pm 2\sqrt{z_1^2 - 1}\sqrt{z_2^2 - 1}\sqrt{z_3^2 - 1}\sqrt{z^2 - 1} - 2; \tag{19}$$

$$z_0 = z_1 z_2 z_3 + z_1\sqrt{z_2^2 - 1}\sqrt{z_3^2 - 1} + z_2\sqrt{z_1^2 - 1}\sqrt{z_3^2 - 1} +$$

$$+ z_3\sqrt{z_1^2 - 1}\sqrt{z_2^2 - 1}. \tag{20}$$

$$K(k) = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}}$$

—is the complete elliptic integral of the first kind.

It is seen from (18) that the ladder diagram of sixth order, in which all internal lines are replaced by δ-functions, has the ability to "split" one Legendre function of the second kind into three.

Multiplying both sides of equality (18) by $(2l+1)P_l(z)$ and summing over l from 0 to ∞, we obtain the following spectral representation for the sum of the product of three Legendre functions of the second kind and a Legendre polynomial:

$$\sum_l (2l+1)\, Q_l(z_1)\, Q_l(z_2)\, Q_l(z_3)\, P_l(z) =$$

$$= \int_{z_0}^{\infty} \frac{d\eta}{\eta - z}\, \frac{2}{\sqrt{\Delta^+(z_1 z_2 z_3 \eta)}}\, K\left(\sqrt{\frac{\Delta^-(z_1 z_2 z_3 \eta)}{\Delta^+(z_1 z_2 z_3 \eta)}}\right), \qquad (21)$$

whence, with the aid of the Watson–Sommerfeld transformation, it is easy to obtain the value of integral (10) given in the text.

In conclusion I express my gratitude to Prof. A. A. Sokolov for his interest in this work.

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Received
10 IX 1962

CITED LITERATURE

[1] T. Regge, *Nuovo Cim.*, **14**, 972 (1953); **18**, 947 (1960).
[2] G. Wanders, *Nuovo Cim.*, **17**, 535 (1961); **18**, 580 (1961).
[3] K. A. Ter-Martirosyan, lecture in the collection *Problems in the Theory of Strong and Weak Interactions of Elementary Particles*, Yerevan, 1961.
[4] W. Zimmerman, *Nuovo Cim.*, **21**, 249 (1691).
[5] A. A. Sokolov, *J. Phys. USSR*, **5**, 231 (1941).
[6] A. A. Sokolov, B. K. Kerimov, *DAN*, **108**, 611 (1956).
[7] M. Froissart, *Nuovo Cim.*, **22**, 395 (1961).
[8] R. M. Muradyan, *DAN*, **131**, 1057 (1960).

Note: Figure translations are in progress. See original paper for figures.