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# MATHEMATICS

1963

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**Abstract**

**Full Text**

## MATHEMATICS

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### ESTIMATES IN THE THEORY OF REGULAR FUNCTIONS OF TWO COMPLEX VARIABLES

*(Presented by Academician V. I. Smirnov on 4 III 1963)*

The problem of obtaining a number of classical estimates for Schur functions, Carathéodory functions, and some others in the case of more than one complex variable was considered by me in papers <sup>(1-3)</sup>, where this problem was solved for the indicated functions regular in bicircular domains. In the present note the estimates mentioned above are established for the same functions, but regular in domains of a new nature. From this note the corresponding results of papers <sup>(1,3)</sup> are obtained as a special case when  $\rho(\theta) \equiv 1$ .

Take the domain  $D_1$  (containing the point  $(0, 0)$ ), bounded by the hypersurface <sup>(4)</sup>:

$$w = r_1(\tau)\xi, \quad z = r_2(\tau)\xi e^{-it}, \quad 0 \leq \tau \leq 1, \quad 0 \leq t \leq 2\pi, \quad (1)$$

and the domain  $D_2$  (containing the point  $(0, 0)$ ), bounded by the hypersurface

$$w = r_1(\tau)\xi e^{-it}, \quad z = r_2(\tau)\xi, \quad 0 \leq \tau \leq 1, \quad 0 \leq t \leq 2\pi,$$

where  $r_1(0) = 0$ ,  $r_1(1) < \infty$ ,  $0 < r_1'(\tau) \leq$  (or  $=$ )  $r_1(\tau)\tau^{-1}$ ,  $0 < \tau \leq 1$ ,

$$r_2(\tau) = R_2 \exp \left( - \int_0^\tau \frac{\tau}{1-\tau} d \ln r_1(\tau) \right)$$

( $R_2$  is a positive constant),  $\xi = \rho(\theta)e^{i\theta}$ ,  $0 \leq \theta \leq 2\pi$ . Here  $\rho(\theta)$  is a real, positive, continuous, periodic function of the argument  $\theta$  with period  $2\pi$ . Note the equality  $r_2(1) = 0$  <sup>(5)</sup>.

The domain containing the point 0 and bounded by the rectifiable Jordan curve  $\xi = \rho(\theta)e^{i\theta}$ ,  $0 \leq \theta \leq 2\pi$ , will be denoted by  $B$ . Let  $\eta = \varphi(\xi)$ ,  $\varphi(0) = 0$ ,  $\varphi'(0) > 0$ , be a function regular in  $B$  and mapping the domain  $B$  univalently onto the disk  $|\eta| < 1$ , and let  $\xi = \psi(\eta)$  be the inverse function.

**Theorem 1.** *If in the domain  $D_1$  the function  $F(w, z)$  ( $F(0, 0) = 0$ ) is regular and  $|F(w, z)| < 1$ , then in  $D_1$*

$$|F(w, z)| \leq \alpha, \quad (2)$$

where

$$\alpha = \begin{cases} \frac{|z|}{r_2(0) \sup_{0 \leq \theta \leq 2\pi} \rho(\theta)} & \text{for } w = 0, \\ |\varphi(wr_1^{-1}(1))| & \text{for } z = 0, \\ |\varphi(\omega \exp(i \arg w))|, \quad \omega = \sup_{0 < \tau < 1} (\tau r_1^{-1}(\tau)|w| + (1 - \tau)r_2^{-1}(\tau)|z|) & \text{for the remaining points of } D_1. \end{cases}$$

**Proof.** Since the domain  $D_1$  is a domain containing the point  $(0, 0)$  and bounded by the hypersurface (1), for any values of the parameters  $\tau, t$ ,  $0 \leq \tau \leq 1$ ,  $-\infty < t < +\infty$ , the point  $(r_1(\tau)\xi, r_2(\tau)\xi e^{-it}) \in D_1$ , if  $\xi \in B$ . Therefore, by virtue of the hypothesis of the theorem, the function  $F(r_1(\tau)\psi(\eta), r_2(\tau)\psi(\eta)e^{-it})$ , for any fixed values of the parameters  $\tau, t$ ,

$0 \leq \tau \leq 1$ ,  $-\infty < t < +\infty$ , as a function of one complex variable  $\eta$ , satisfies the conditions of Schwarz' s lemma in the circle  $|\eta| < 1$ . Consequently,

$$|F(r_1(\tau)\psi(\eta), r_2(\tau)\psi(\eta)e^{-it})| \leq |\eta| \quad (0 < |\eta| < 1)$$

or

$$|F(r_1(\tau)\zeta, r_2(\tau)\zeta e^{-it})| \leq |\varphi(\zeta)| \quad (\zeta = \psi(\eta)). \quad (3)$$

Let now  $(w_0, z_0)$  be an arbitrary point of the domain  $D_1$ , distinct from the points  $(w, 0)$ ,  $(0, z)$  of this domain, and let

$$w = r_1(\tau)\lambda_0\rho(\theta)e^{i\theta}, \quad z = r_2(\tau)\lambda_0\rho(\theta)e^{i(\theta-t)}, \quad (4)$$

$$0 \leq \tau \leq 1, \quad 0 < \lambda_0 < 1, \quad 0 \leq \theta, t \leq 2\pi,$$

be a hypersurface containing the point  $(w_0, z_0)$ . Such a hypersurface exists, since, as is easy to show, there exist  $\theta_0, t_0, \tau_0, \lambda'_0$ ,  $0 \leq \theta_0 \leq 2\pi$ ,  $0 \leq t_0 \leq 2\pi$ ,  $0 < \tau_0 < 1$ ,  $0 < \lambda'_0 < 1$ , for which

$$w_0 = r_1(\tau_0)\lambda_0\rho(\theta_0)e^{i\theta_0}, \quad z_0 = r_2(\tau_0)\lambda_0\rho(\theta_0)e^{i(\theta_0-t_0)}.$$

By virtue of estimate (3), for the points of the hypersurface (4) we have

$$|F(w, z)| \leq |\varphi(\zeta)| \quad (\zeta = \lambda_0 \rho(\theta) e^{i\theta}),$$

whence, in particular, at the point  $(w_0, z_0)$ ,

$$|F(w_0, z_0)| \leq \varphi|\zeta_0| \quad (\zeta_0 = \lambda_0 \rho(\theta_0) e^{i\theta_0}). \quad (5)$$

We shall show that

$$\zeta_0 = \left( \sup_{0 < \tau < 1} (\tau r_1^{-1}(\tau)|w_0| + (1 - \tau)r_2^{-1}(\tau)|z_0|) \right) e^{i \arg w_0}. \quad (6)$$

Indeed, from the preceding argument it is obvious that  $\arg \zeta_0 = \arg w_0$ . It remains to show that

$$|\zeta_0| = \sup_{0 < \tau < 1} (\tau r_1^{-1}(\tau)|w_0| + (1 - \tau)r_2^{-1}(\tau)|z_0|). \quad (7)$$

For  $0 < \tau < 1$  we have

$$\tau r_1^{-1}(\tau)|w_0| + (1 - \tau)r_2^{-1}(\tau)|z_0| = (\tau r_1^{-1}(\tau)r_1(\tau_0) + (1 - \tau)r_2^{-1}(\tau)r_2(\tau_0)) |\zeta_0|. \quad (8)$$

Since for the function  $\lambda(\tau) \equiv \tau_1 r_1^{-1}(\tau_1)r_1(\tau) + (1 - \tau_1)r_2^{-1}(\tau_1)r_2(\tau)$ , where  $\tau_1$  is arbitrary in the interval  $0 < \tau < 1$ , on the segment  $0 \leq \tau \leq 1$  the inequality  $\lambda(\tau) \leq 1$  holds <sup>(5)</sup>, it follows that

$$\tau r_1^{-1}(\tau)r_1(\tau_0) + (1 - \tau)r_2^{-1}(\tau)r_2(\tau_0) \leq 1 \quad (0 < \tau < 1). \quad (9)$$

From relations (8) and (9) it follows that

$$\tau r_1^{-1}(\tau)|w_0| + (1 - \tau)r_2^{-1}(\tau)|z_0| \leq |\zeta_0| \quad (0 < \tau < 1),$$

whence

$$|\zeta_0| \geq \sup_{0 < \tau < 1} (\tau r_1^{-1}(\tau)|w_0| + (1 - \tau)r_2^{-1}(\tau)|z_0|).$$

On the other hand,

$$\begin{aligned}
 |\zeta_0| &= \tau_0 r_1^{-1}(\tau_0) r_1(\tau_0) |\zeta_0| + (1 - \tau_0) r_2^{-1}(\tau_0) r_2(\tau_0) |\zeta_0 e^{-it_0}| \\
 &= \tau_0 r_1^{-1}(\tau_0) |w_0| + (1 - \tau_0) r_2^{-1}(\tau_0) |z_0| \\
 &\leq \sup_{0 < \tau < 1} (\tau r_1^{-1}(\tau) |w_0| + (1 - \tau) r_2^{-1}(\tau) |z_0|).
 \end{aligned}$$

Thus, we have (7). Hence, equality (6) has been established.

Denote

$$\omega_0 = \sup_{0 < \tau < 1} (\tau r_1^{-1}(\tau) |w_0| + (1 - \tau) r_2^{-1}(\tau) |z_0|);$$

then, taking equality (6) into account, estimate (5) takes the form

$$|F(w_0, z_0)| \leq |\varphi(\omega_0 \exp(i \arg w_0))|.$$

But the point  $(w_0, z_0)$  is an arbitrary point of the domain  $D_1$ , distinct from the points  $(w, 0)$ ,  $(0, z)$  of this domain, and therefore for all points  $(w, z) \in D_1$ , except the points  $(w, 0)$ ,  $(0, z)$  of this domain, we have the estimate

$$|F(w, z)| \leq |\varphi(\omega \exp(i \arg w))|,$$

where

$$\omega = \sup_{0 < \tau < 1} (\tau r_1^{-1}(\tau) |w| + (1 - \tau) r_2^{-1}(\tau) |z|).$$

Let us find estimates for  $|F(w, z)|$  for the points  $(w, 0)$  and  $(0, z)$  of the domain  $D_1$ . From inequality (3), for  $\tau = 1$  we have

$$|F(r_1(1)\xi, 0)| \leq |\varphi(\xi)|$$

or

$$|F(w, 0)| \leq |\varphi(w r_1^{-1}(1))|.$$

Next, from the parametric representation of the boundary of the domain  $D_1$ , for  $\tau = 0$  we have  $w = 0$ ,  $z = r_2(0) \rho(\theta) e^{i\theta} e^{-it}$ ,  $0 \leq \theta, t \leq 2\pi$ , i.e. we obtain a family of circles ( $\theta$  is a parameter) in the plane  $w = 0$  with center at the point  $z = 0$ . In each of the disks bounded by these circles, the function  $F(0, z)$ , by virtue of the condition of the present theorem, satisfies Schwarz' s lemma. Therefore, in the disk  $|z| < r_2(0) \sup_{0 \leq \theta \leq 2\pi} \rho(\theta)$  we have

$$|F(0, z)| \leq \frac{|z|}{r_2(0) \cdot \sup_{0 \leq \theta \leq 2\pi} \rho(\theta)}.$$

Finally, introducing the notation for  $\alpha$  indicated in the statement of the present theorem, we obtain the required estimate (2). The theorem is proved.

Theorems 2-5 are proved in the same way as Theorem 1, but using the corresponding propositions for functions of one complex variable (6-8).

**Theorem 2.** If in the domain  $D_1$  the function  $F(w, z)$  ( $F(0, 0)$  real) is regular and  $\operatorname{Re} F(w, z) > 0$ , then in  $D_1$  the following estimates hold:

$$F(0, 0)(1 - \alpha)(1 + \alpha)^{-1} \leq \operatorname{Re} F(w, z) \leq F(0, 0)(1 + \alpha)(1 - \alpha)^{-1},$$

$$|\operatorname{Im} F(w, z)| \leq 2\alpha F(0, 0)(1 - \alpha^2)^{-1},$$

$$F(0, 0)(1 - \alpha)(1 + \alpha)^{-1} \leq |F(w, z)| \leq F(0, 0)(1 + \alpha)(1 - \alpha)^{-1}.$$

**Theorem 3.** Let in the domain  $D_1$  the function  $F(w, z)$  ( $F(0, 0) = 0$ ) be regular and  $|\operatorname{Re} F(w, z)| < 1$ . Then in  $D_1$  the estimates are valid:

$$|\operatorname{Re} F(w, z)| \leq 4\pi^{-1} \operatorname{arctg} \alpha,$$

$$|\operatorname{Im} F(w, z)| \leq \frac{2}{\pi} \ln \frac{1 + \alpha}{1 - \alpha}, \quad |F(w, z)| \leq \frac{2}{\pi} \ln \frac{1 + \alpha}{1 - \alpha}.$$

**Theorem 4.** If in the domain  $D_1$  the function  $F(w, z)$  is regular and  $|F(w, z)| < 1$ , then in  $D_1$

$$|wF'_w(w, z) + zF'_z(w, z)| \leq (1 - |F(w, z)|^2)\beta(1 - \alpha^2)^{-1},$$

where

$$\beta = \begin{cases} \frac{|z|}{r_2(0) \sup_{0 \leq \theta \leq 2\pi} \rho(\theta)} & \text{for } w = 0, \\ |\varphi'(wr_1^{-1}(1))| |w| r_1^{-1}(1) & \text{for } z = 0, \\ |\varphi'(\omega \exp(i \arg w))| \omega & \text{for the remaining points of } D_1. \end{cases}$$

**Corollary.** Under the assumptions of Theorem 4, in the domain  $D_1$  the estimate

$$|wF'_w(w, z) + zF'_z(w, z)| \leq \beta(1 - \alpha^2)^{-1}.$$

**Theorem 5.** If the function  $F(w, z)$ , regular in the domain  $D_1$ , has a nonnegative real part in it, then in  $D_1$

$$|wF'_w(w, z) + zF'_z(w, z)| \leq 2\beta(1 - \alpha^2)^{-1} \operatorname{Re} F(w, z).$$

From Theorems 1, 2, and 5 it follows that:

**Theorem 6.** If in the domain  $D_1$  the function  $F(w, z)$  is regular and  $\operatorname{Re} F(w, z) < u$ , then in  $D_1$  the estimates

$$u - (u - \operatorname{Re} F(0, 0)) \frac{1 + \alpha}{1 - \alpha} \leq \operatorname{Re} F(w, z) \leq u - (u - \operatorname{Re} F(0, 0)) \frac{1 - \alpha}{1 + \alpha},$$

$$|\operatorname{Im} F(w, z) - \operatorname{Im} F(0, 0)| \leq 2\alpha(u - \operatorname{Re} F(0, 0))(1 - \alpha^2)^{-1},$$

$$|F(w, z) - F(0, 0)| \leq 2\alpha(u - \operatorname{Re} F(0, 0))(1 - \alpha)^{-1},$$

$$|wF'_w(w, z) + zF'_z(w, z)| \leq 2\beta(u - \operatorname{Re} F(w, z))(1 - \alpha^2)^{-1}$$

hold.

**Remark 1.** As is seen from the proof of Theorem 1, the domain  $D_1$  has the property that, together with each point  $(w_0, z_0)$ , it also contains all points with coordinates  $w = w_0$ ,  $z = z_0 e^{is}$ , where  $s$  is an arbitrary real number. Consequently<sup>(9)</sup>, the domain  $D_1$  is a circular domain with plane of symmetry  $z = 0$ .

**Remark 2.** Theorems 1-6, with the corresponding changes in the notation for  $\alpha$  and  $\beta$ , also hold in the case of the domain  $D_2$ . The proofs are the same as in the case of the domain  $D_1$ .\*\*

**Remark 3.** For the corresponding special classes of domains in the space of  $n$  complex variables, all arguments are carried out in a completely analogous way.

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Received  
2 II 1963

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\* Here  $\varphi'(\xi)$  is meant.

\*\* Just as in the case of the domain  $D_1$ , it is established that the domain  $D_2$  is a circular domain with plane of symmetry  $w = 0$ .

*Note: Figure translations are in progress. See original paper for figures.*

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