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Abstract

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CLOSED-GRAPH THEOREMS FOR UNIFORM SPACES

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V. Pták ([1,2]), analyzing the well-known Banach theorem on open mappings ([3]), introduced a class of locally convex spaces, called by him B-complete and characterized by the fact that every "almost open" continuous linear mapping of them onto any locally convex space is open. A. and W. Robertson ([4]) extended to B-complete spaces also Banach' s closed-graph theorem ([3]), and then Pták ([5]) embraced both properties by a single theorem, which is essentially a closed-graph theorem for multivalued linear mappings of B-complete spaces. Pták used methods of the duality theory of locally convex spaces. But J. Kelley ([6]) showed that B-completeness of a locally convex space is equivalent to a certain property of weakened completeness of the space of its closed subsets, endowed with the "Hausdorff uniformity," and, using this, obtained a significant part of the results of Pták and the Robertsons without the aid of duality theory. This made it possible, by a small modification of Kelley' s arguments, to extend the theory of B-completeness also to non-locally convex topological linear spaces ([7]).

The purpose of the present note is to transfer the theory of B-completeness to uniform spaces.

1. Let E and F be nonempty sets. A **multivalued mapping** (in what follows, more often simply a **mapping**) from E to F is a mapping f assigning to each element $x \in E$ a set $fx \subset F$. The **domain of definition** of f is the set D_f of those $x \in E$ for which the image fx is nonempty. f is called **single-valued** if fx is a singleton for every $x \in D_f$. f is called a **mapping** of E into F (and not "from E into F") if $D_f = E$.

Let f be a mapping from E to F. The mapping f^{-1} from F to E, **inverse** to f, is defined by the condition that the relations $x \in f^{-1}y$ and $y \in fx$ are equivalent. The **graph** of f is the set $\{(x, y) \in E \times F : y \in fx\}$. By $\tilde{f}$ is meant $f \times f$, i.e., the mapping from $E \times E$ to $F \times F$ given by the formula

$$\tilde{f}(x_1, x_2) = \{(y_1, y_2) \in F_1 \times F_2 :\ y_1 \in fx_1,\ y_2 \in fx_2\}.$$

$R \subset E \times E$ (as well as a mapping from E to E with graph R) is called a **partial equivalence** if $R^2 = R^{-1} = R$, or, equivalently, $R^{-1}R = R$.

2. The uniform space obtained by endowing the set E with a uniformity u (which we shall identify with the collection of its entourages) is denoted by (E, u); (E, u) is not assumed to be separated.

We shall say that a mapping f from a uniform space (E, u) to a uniform space (F, v) is **uniformly continuous** if for every $V \in v$ there exists a $U \in u$ such that $Ux \subset f^{-1}Vy$ for all $y \in F$ and all $x \in f^{-1}y$; **uniformly almost continuous** if for every $V \in v$ there exists a $U \in u$ such that $Ux \subset \overline{f^{-1}Vy}$ for all $y \in F$ and all $x \in f^{-1}y$ (where the bar denotes closure); **uniformly open** if f^{-1} is uniformly continuous; **uniformly almost open** if f^{-1} is uniformly almost continuous; **uniformly**

homeomorphic if f is uniformly continuous and uniformly open; it has a closed graph if the graph of f is closed in the uniform space $(E, u) \times (F, v)$.

3. Let f be a mapping from a uniform space (E, u) into a set F. We shall say that f is **regular** if $\tilde{f}(u)$ is a uniformity on $f(E)$, and f is a uniformly open mapping from (E, u) onto $(f(E), \tilde{f}(u))$. We shall say that the mapping f from (E, u) into (F, v) is **biregular** if f is a regular mapping from (E, u) into F, and f^{-1} is a mapping from (F, v) into E.

Examples. a) A one-to-one mapping from (E, u) into F (respectively from (E, u) into (F, v)) is regular (respectively biregular).
b) A mapping inverse to a single-valued mapping from F into (E, u) is regular.
c) A single-valued uniformly homeomorphic mapping from (E, u) into (F, v) is biregular.
d) Projections of a product of uniform spaces onto these spaces are biregular.
e) An algebraic homomorphism from a topological group G into a group H (respectively into a topological group H) is regular (respectively biregular).
f) The canonical mapping of a topological group G, endowed with the right uniformity, onto the homogeneous space G/H of left cosets modulo the subgroup H is biregular.

Theorem 1. *If f is a regular mapping with closed graph from (E, u) into (F, v), then the images fx_1, fx_2 of points $x_1, x_2 \in E$, as well as the inverse images $f^{-1}y_1, f^{-1}y_2$ of points $y_1, y_2 \in F$, either coincide or do not intersect.*

4. Let (E, u) be a uniform space. Denote by $\mathcal{E}_u$ the uniform space whose elements are all nonempty subsets of the product $E \times E$, and whose uniformity is given by the base of entourages $\{(M, N) : M \subset UN,\ N \subset UM\}$, where U ranges over u.

A B**-net** for (E, u) will mean a net (generalized sequence) $\{W_\alpha\}_{\alpha \in A}$ in $\mathcal{E}_u$ satisfying the following conditions: 1) $\{W_\alpha\}$ is a fundamental system of symmetric entourages of some uniformity on some set $E_0 \subset E$; 2) the partial equivalence

$$R = \bigcap_{\alpha \in A} W_\alpha$$

is regular; 3)

$$\bigcap_{\alpha \in A,\, U \in u} UW_\alpha = R;$$

4) $\{W_\alpha\}_{\alpha \in A}$ is a Cauchy net in $\mathcal{E}_u$ (i.e., for every $U \in u$ there exists $\alpha_U \in A$ such that

$$W_{\alpha'} \subset UW_\alpha$$

for all $\alpha, \alpha' \geq \alpha_U$).

We shall call (E, u) B**-complete** if every B-net $\{W_\alpha\}_{\alpha \in A}$ in $\mathcal{E}_u$ converges (then it converges to R, i.e., for every $U \in u$ there exists $\alpha_U \in A$ such that

$$W_\alpha \subset UR$$

for all $\alpha \geq \alpha_U$).

Theorem 2. *The following assertions about a uniform space* (E, u) *are equivalent:*

1) (E, u) is B-complete;
2) every uniformly nearly continuous biregular mapping with closed graph of any uniform space (F, v) into (E, u) is uniformly continuous;
3) every uniformly nearly open biregular mapping with closed graph from (E, u) onto any uniform space (F, v) is uniformly open;
4) every single-valued uniformly nearly open regular mapping with closed graph from (E, u) onto any uniform space (F, v) is uniformly open.
5. A B'**-net** for (E, u) will mean any B-net for which $E_0 = E$. We shall call the space (E, u) B'**-complete** if every B'-net in $\mathcal{E}_u$ converges. Obviously, every B-complete space is B'-complete.

Theorem 2′. *The following assertions about a uniform space* (E, u) *are equivalent:*

1′) (E, u) is B'-complete;

2′) every uniformly nearly continuous biregular mapping-

with closed graph from any uniform space (F, v) onto (E, u) is uniformly continuous;

3′) every uniformly nearly open biregular mapping with closed graph from the space (E, u) onto any uniform space (F, v) is uniformly open;

4′) every single-valued uniformly nearly open regular mapping with closed graph from the space (E, u) onto any uniform space (F, v) is uniformly open;

5′) every single-valued uniformly nearly continuous regular mapping with closed graph from any uniform space (F, v) onto the quotient space $(E/R, u/R)$ of the space (E, u) by a regular equivalence R is uniformly continuous.

6. Let $\mathfrak{p}(E)$ be the set of all nonempty subsets of the uniform space (E, u). The **Hausdorff uniformity** in $\mathfrak{p}(E)$ is the uniformity $\mathfrak{p}(u)$ defined by the base of entourages

$$\{(A, B) : A \subset U(B),\ B \subset U(A)\},$$

where U runs through u. (E, u) is called **ultracomplete** if $(\mathfrak{p}(E), \mathfrak{p}(u))$ is complete.

(E, u) is ultracomplete if and only if $\mathcal{E}_u$ is complete. Thus every ultracomplete space is B-complete, and hence possesses the "closed graph properties" of Theorems 2–2′.

Theorem 3. The quotient space $(E/R; u/R)$ of an ultracomplete (respectively B-complete, B'-complete) space (E, u) by a regular equivalence R is ultracomplete (respectively B-complete, B'-complete).

As is known, complete metrizable uniform spaces are ultracomplete. Another example of ultracomplete spaces is provided by uniformly locally bicompact spaces. Necessary and sufficient conditions for ultracompleteness of a uniform space were recently found by Isbell ([8]).

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Note: Figure translations are in progress. See original paper for figures.