# SPLIT LINEAR GROUPS

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## Abstract

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## MATHEMATICS

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# SPLIT LINEAR GROUPS

*(Presented by Academician A. I. Mal'  tsev on 24 I 1963)*

Let $P$ be a perfect field, and let $L_n(P)$ be the full linear group of degree $n$ over $P$. Then every matrix $g \in L_n(P)$ has a unique Jordan decomposition: $g = g_s \cdot g_u = g_u \cdot g_s$, where $g_s$ is semisimple and $g_u$ is a unipotent matrix.

All groups considered below are linear.

**Definition 1.** A group $G$ is called **split** if, together with every matrix $g \in G$, we have $g_s, g_u \in G$.

This definition was proposed by A. I. Mal'  tsev in ($^1$), where complex and real solvable split Lie algebras (groups) were first introduced and studied.

It is known ($^2$) that every algebraic group over a perfect field is split. At the same time, for split groups there are no such powerful methods of study as for algebraic groups. In essence, the only method for studying split linear groups is their reduction to algebraic groups.

Relying on A. Borel'  s fundamental results ($^2$) on algebraic groups, Togo has recently ($^{3-5}$) transferred Borel'  s main results to certain classes of connected split linear groups over an algebraically closed field. In these works, primary attention is given to the consideration of solvable and nilpotent groups.

In the present article arbitrary (not necessarily connected) locally nilpotent and solvable split groups over a perfect field are studied. Along the way, we establish the structure of arbitrary locally nilpotent linear groups over a perfect field in a simpler way, and in a way that more deeply reveals the essence, than in ($^8$). In conclusion, disconnected split solvable Lie groups are studied, thereby generalizing the above-mentioned results of A. I. Mal'  tsev.

The corresponding results of Togo follow from the results of our article. In the proof we make essential use of the author'  s results on algebraic groups ($^6$,$^7$ ).

The terminology of the article for the most part coincides with the terminology adopted in ($^2$,$^3$ ). All topological notions, unless otherwise specified, refer to the Zariski topology. By $G_s$ ($G_u$) we denote the set of all semisimple (respectively unipotent) matrices of the group $G$.

From Theorem 1 in ($^8$) it follows easily that

**Proposition 1.** *Let $G$ be a locally nilpotent group; then $G_u$ and $G_s$ are invariant subgroups in $G$.*

**Theorem 1.** *Every split locally nilpotent group $G$ can be represented in the form of the direct product $G = G_s \cdot G_u$.*

The proof follows from Proposition 1 and the splitness of $G$. Theorem 1 generalizes Theorem 4 of ([8]).

Since the intersection of split groups is, obviously, a split group, one can speak of the smallest split group $G^R$ containing a given group $G$. In view of the splitness of algebraic···

groups, $G^R \subseteq \bar{G}$, where $\bar{G}$ is the closure of $G$ in the Zariski topology. Hence follows

**Proposition 2.** *If $G$ is a locally nilpotent group, then $G^R$ is also locally nilpotent.*

We have shown that the closure, in the Zariski topology, of a locally nilpotent non-nilpotent group is not locally nilpotent; therefore every maximal locally nilpotent subgroup $L_n(P)$ is split, but not algebraic.

From Theorem 1 and Proposition 2 there follows the main theorem from (8).

**Theorem 2.** *Every locally nilpotent group $G$ is contained in a uniquely determined locally nilpotent group $G^R = G_1^R \cdot G_u^R$.*

We also note the following assertion, obtained by A. Borel ([2]) for algebraic groups.

**Proposition 3.** *Let $G$ be a split nilpotent group, $G_0$ its connected component. Then $G_{0s}$ belongs to the center of $G$. Every semisimple matrix $g \in G$ centralizes $G_0$.*

**Proof.** By Theorem 1, $G = G_s \cdot G_u$,

$$G_0 = G_{0s} \cdot G_{0u}, \tag{1}$$

where $G_{0s} \subseteq G_s$, $G_{0u} \subseteq G_u$. In every completely reducible nilpotent group the center has finite index ([9], Ch. 3), and since the group $G_s$ is completely reducible, $G_{0s}$ lies in the center of $G_s$, and by virtue of (1) also in the center of $G$. Hence the last assertion of Proposition 3 follows as well.

**Definition 2.** A solvable group $G$ is called an $S$-group if for every maximal $d$-subgroup (i.e., a subgroup consisting of semisimple matrices) $D \subset G$, $G = DG_u$.

The following questions naturally arise: 1) when is a split solvable group an $S$-group? 2) when is a solvable $S$-group split? Even in the case of connected groups, as examples show (see ([4]), Examples 1 and 2), the answer to these questions is, generally speaking, negative.

Let $G$ be a solvable split connected group, $\bar G$ the closure of $G$ in the Zariski topology.

A **torus** is a connected abelian $d$-group.

**Proposition 4.** *$G$ has maximal tori. If $Q$ is a maximal abelian $d$-subgroup of $G$, then $\bar Q$ is a maximal torus of $\bar G$.*

**Proof.** It is clear that the centralizer $Z^{\bar G}(\bar Q) = \overline{Z^G(Q)}$, with $Z^G(Q)$ a split group. Then $Z^G(Q) = QZ^G_u$, and hence $Z^{\bar G}(\bar Q) = \bar Q\,\overline{Z^G_k}$, which proves Proposition 4.

**Proposition 5.** $(\bar G)_u = \overline{(G_u)}$.

**Proof.** $\overline{(G_u)}$ is a normal divisor of $\bar G$. Let $f$ be a rational representation of $\bar G$ with kernel $\overline{(G_u)}$; then $f(\bar G) = \bar G'$, $f(G) = G'$, where $G'$ is an abelian $d$-group. There exists an abelian $d$-subgroup $Q^* \subset \bar G$ such that $f(Q^*) = G'$, whence $G \subset Q^* \cdot \overline{(G_u)}$; consequently,

$$\bar G = \bar Q^* \cdot \overline{(G_u)},$$

and thus Proposition 4 is proved.

Let $c^1\bar G = [\bar G, \bar G], \dots,\ c^i\bar G = [c^{i-1}\bar G, \bar G]$ and $c^\infty\bar G = \bigcap_i c^i\bar G$. Since $c_i\bar G$ are algebraic groups, $c^\infty\bar G$ is also algebraic; moreover, in view of the termination of descending chains of algebraic varieties, the factor group $\bar G/c^\infty\bar G$ is nilpotent.

**Definition 3.** A group $G$ is called a $c^\infty$-group over $P$ if $c^\infty\bar G \subset G$, where $\bar G$ is the closure of $G$ over $P$.

It is clear that every algebraic group is a $c^\infty$-group. It can be shown that a connected complex Lie group will also be a $c^\infty$-group.

**Theorem 3.** *A connected split solvable $c^\infty$-group $G$ is an $S$-group.*

**Proof.** Let $\bar f$ be a rational representation of $\bar G$ with kernel $c^\infty\bar G$, and let $f$ be the restriction of $\bar f$ to $G$; then $f(G) = G'$, where $G'$ is a nilpotent splittable group; consequently, by Theorem 1, $G' = G'_s \cdot G'_u$. If $G'_u = (e)$, then $c^\infty\bar G = G_u$, and hence, by Proposition 5, we obtain $G_u = \overline{(G_u)}$. Let $\bar G = TG_u$, where $T$ is a maximal torus of $\bar G$; then $G = T'G_u$, where $T' = T \cap G$. If, however, $G'_u \neq (e)$, consider $H = f^{-1}(G'_s)$. The group $H$ is splittable, since if $h = h_s \cdot h_u$, then

$$f(h) = f(h_s) \cdot f(h_u) = f(h_s) \subset G'_s,$$

i.e. $h_s \in H$. Clearly, $H_u = c^\infty\bar G$, hence $\overline{(H_u)} = H_u$, whence, as above, $H = T \cdot H_u$, where $T$ is an arbitrary maximal torus in $G$; therefore

$$G = H \cdot G_u = T \cdot H_u \cdot G_u = T \cdot G_u.$$

If a solvable connected group $G$ is an $S$-group, then the question of conjugacy of complements is always settled affirmatively in the following sense.

**Theorem 4.** If $G = D_1 G_u$ and $G = D_2 G_u$, then $D_1$ and $D_2$ are conjugate by a unipotent matrix from the maximal solvable connected group $R \supseteq G$.

We now pass to the consideration of disconnected solvable splittable groups. In this case an adequate study is possible only over a field of characteristic zero (see (7)); henceforth $P$ is a field of characteristic zero.

**Theorem 5.** If $G$ is a solvable algebraic group, then all maximal $d$-subgroups in $G$ are conjugate by unipotent matrices from $c^\infty G$.

**Proof.** Let $f$ be a rational representation of $G$ with kernel $c^\infty G$. Let

$$f(G) = G' = G'_s \cdot G'_u,$$

and let $Q_1$ and $Q_2$ be maximal $d$-subgroups in $G$. If $G'_u = (e)$, then $c^\infty G \supseteq G_u$, and the theorem is true, since in (7) the conjugacy of $Q_1$ and $Q_2$ by means of a matrix from $G_u$ is proved. If $G'_u \neq (e)$, consider $f^{-1}(G'_s) = H$. Clearly $Q_1, Q_2 \subset H$; then $H = Q_1 H_u = Q_2 H_u$, where $H_u$ is the unipotent part of the group $c^\infty G$. It follows that $H$ is an algebraic group, and by Theorem A from (7), $Q_1$ and $Q_2$ are conjugate by means of a matrix from $H_u \subset c^\infty G$, which proves Theorem 5.

**Theorem 6.** A solvable $c^\infty$-group $G$ is splittable if and only if it is an $S$-group.

**Proof.** Let $D_1$ and $D_2$ be maximal $d$-subgroups of $G$, and let $D_1^*, D_2^*$ be maximal $d$-subgroups of $\bar{G}$ containing them, respectively. By Theorem 5, there exists $t \in c^\infty \bar{G}$ such that $t D_1^* t^{-1} = D_2^*$. Since $t \in G$, we have $t D_1 t^{-1} = D_2$, which proves the last assertion of Theorem 6.

1) Let $G$ be a splittable solvable $c^\infty$-group; then it is an $S$-group. The proof is analogous to the proof of Theorem 3.

2) Let $G$ be an $S$-group. We show that together with $g \in G$ we have $g_s, g_u \in G$. There exists a maximal $d$-subgroup $D_s \subset \bar{G}$ such that $g_s \in D_s$. Let $D$ be an arbitrary maximal $d$-subgroup of $G$; then $D \subseteq D^*$, where $D^*$ is a maximal $d$-subgroup of $\bar{G}$. From Theorem 5 it follows that there exists $r \in c^\infty \bar{G} \subset G$ such that $r D^* r^{-1} = D_s$; consequently,

$$D_s \supset r D r^{-1} = D_r.$$

Since $G$ is an $S$-group, $\bar{G} = D_r \cdot G_u$. Hence $g = q \cdot t = g_s \cdot g_u$, and

$$g_s^{-1} q = g_{ut}^{-1} = e,$$

i.e. $g_s = q \in G$, which completely proves Theorem 6.

In conclusion we study splittable solvable Lie groups. First, we note that every connected complex Lie group $G$ is a $c^\infty$-group. Indeed, let $L(G)$ be the Lie algebra of the group $G$, and let $L(\bar{G})$ be the Lie algebra of the group $\bar{G}$. As shown in (10), Ch. 2, Theorem 13, the derived algebra $D^1L(\bar{G}) = D^1L(G) \subset L(G)$. Since the commutant $K = D^1\bar{G}$ is a connected group, it follows that $D^1\bar{G} \subset G$.

Using the same argument, one can show that every connected solvable real Lie group is a $c^\infty$-group; consequently, we have

**Proposition 6.** A connected solvable Lie group is a $c^\infty$-group. From Theorem 6 and Proposition 6 there follows

**Theorem 7** ([1]). A connected solvable Lie group $G$ is an $S$-group if and only if it is split. All maximal tori of a split group $G$ are conjugate by matrices from $G_u$.

**Proposition 7.** A unipotent Lie group is algebraic if and only if it is connected in the Euclidean topology.

**Proof.** 1) If $U$ is a unipotent algebraic group, then it is connected according to Proposition 14 of ([11]), Ch. 5, § 3. 2) Now let $U$ be a connected unipotent Lie group. By $L(U)$ denote the Lie algebra of the group $U$. If $X_1, X_2, \dots, X_m$ is a basis in $L(U)$, then the smallest algebraic group $U^*$, generated by $\exp aX_1$, $\exp aX_2, \dots, \exp aX_m$, where $a \in P$, is connected in the Zariski topology, and its Lie algebra coincides with $L(U)$, whence it follows that $U^* = U$.

From Theorem A of ([7]) and Proposition 7 there follows the following assertion, generalizing Theorem 7.

**Theorem 8.** Let $G$ be a solvable (not connected) Lie group whose unipotent part $G_u$ is connected in the Euclidean topology. Then $G$ is split if and only if it is an $S$-group. All maximal $d$-subgroups of a split group $G$ are conjugate by matrices from $G_u$.

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*Note: Figure translations are in progress. See original paper for figures.*

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