CYBERNETICS AND CONTROL THEORY

B. S. VERKHOVSKII

1963

SovietRxiv

Abstract

Full Text

CYBERNETICS AND CONTROL THEORY

B. S. VERKHOVSKII

MULTIDIMENSIONAL LINEAR PROGRAMMING PROBLEMS OF THE TRANSPORTATION TYPE

(Presented by Academician A. I. Berg, 31 X 1962)

Elements of an s-dimensional matrix are considered, each cell of which is determined by s coordinates (indices) $i_1, \dots, i_s$, where the k-th coordinate takes the values $i_k = 1, \dots, n_k$; all $n_k \geq 2$.

Let M be the set of numbers of all coordinates; R a family of subsets of the set M; $M_j \in R$ the j-th subset, and $|M_j| = m(j)$ the number of its elements; $j = 1, \dots, t$.

Consider an arbitrary cell. Denote by $j(i_1, \dots, i_s)$ those coordinates of the cell whose numbers are not elements of M_j. Let, further,

$$\sum_{k \in M_j}$$

denote summation over all values of all those coordinates whose numbers belong to M_j. All elements of the matrix $\{p_{i_1 \dots i_s}\}$ are given.

We consider the problem of finding all elements of the matrix $\{x_{i_1 \dots i_s}\}$ satisfying the conditions

$$\sum_{k \in M_j} x_{i_1 \dots i_s} = a_{j(i_1 \dots i_s)}, \qquad x_{i_1 \dots i_s} \geq 0 \tag{1}$$

and giving a minimum to the functional

$$L = \sum_{k \in M} p_{i_1 \dots i_s} x_{i_1 \dots i_s}. \tag{2}$$

All $a_{j(i_1, \dots, i_s)} \geq 0$ are prescribed.

Remark. In the case when $1 \leq m(1) = \dots = m(t) \leq s-1$; $t = C_s^{m(j)}$, we obtain the symmetric multi-index transportation problem of type T_m ([3]).

Theorem 1. *In order that the problem have a feasible solution, it is necessary that the following compatibility conditions be satisfied:*

$$\sum_{k\in(M_{j_1}\cup M_{j_2})\setminus M_{j_1}} a_{j_1(i_1,\dots,i_s)} = \sum_{k\in(M_{j_1}\cup M_{j_2})\setminus M_{j_2}} a_{j_2(i_1,\dots,i_s)}, \qquad \text{where } 1\le j_1<j_2\le t. \tag{3}$$

Proof.

$$\sum_{k\in(M_{j_1}\cup M_{j_2})\setminus M_{j_1}} a_{j_1(i_1,\dots,i_s)} = \sum_{k\in(M_{j_1}\cup M_{j_2})\setminus M_{j_1}} \sum_{k\in M_{j_1}} x_{i_1\dots i_s} =$$

$$= \sum_{k\in M_{j_1}\cup M_{j_2}} x_{i_1\dots i_s} = \sum_{k\in(M_{j_1}\cup M_{j_2})\setminus M_{j_2}} a_{j_2(i_1,\dots,i_s)},$$

which was required to be proved.

Further, in Theorems 2-5, cases are indicated in which problem (1)–(2) reduces to one or several simpler ones, and constructive proofs are given.

Theorem 2. *If there exist M_{j_1} and M_{j_2} such that $M_{j_1}\subset M_{j_2}$ $(j_1\ne j_2)$, then the conditions*

$$\sum_{k\in M_{j_2}} x_{i_1\dots i_s} = a_{j_2(i_1,\dots,i_s)}$$

are dependent.

Proof. All the conditions

$$\sum_{k\in M_{j_2}} x_{i_1\dots i_s} = a_{j_2(i_1,\dots,i_s)}$$

can be obtained from the conditions

$$\sum_{k\in M_{j_1}} x_{i_1\dots i_s} = a_{j_1(i_1,\dots,i_s)},$$

if the latter are summed over the indices whose numbers belong to the set $M_{j_2}\setminus M_{j_1}$.

Theorem 3. *If the union of all M_j does not form a cover of M, then problem (1)–(2) decomposes into several problems, the number of indices in each of which is equal to*

$$\left|\bigcup_{j=1}^{t} M_j\right|.$$

Proof. The union of all M_j means the totality of the numbers of all those indices with respect to each of which summation is performed at least once in the conditions (1). If

$$\left|\bigcup_{j=1}^{t} M_j\right| < s,$$

then there exist indices with respect to which no summation is performed. Let

$$M \setminus \bigcup_{j=1}^{t} M_j = \{1, 2, \dots, r\}.$$

Then the original problem decomposes into

$$\prod_{k=1}^{r} n_k$$

problems, each of which does not depend on the first r coordinates.

Theorem 4. *If there exists a subset $\widetilde{M}$ such that $|\widetilde{M}| = r \geqslant 2$ and $|M_j \cap \widetilde{M}| = r$, for all $j = 1, \dots, t$, but*

$$\left|\bigcap_{j=1}^{t} M_j\right| = 0,$$

then problem (1)–(2) reduces to a problem whose number of indices is $s - r + 1$.

Proof. Let $\widetilde{M} = \{1, 2, \dots, r\}$. Then, instead of the indices $i_1, \dots, i_r$, introduce the unfolding index

$$l = i_1 + n_1(i_2 - 1) + n_1 n_2 (i_3 - 1) + \dots + n_1 n_2 \cdots n_{r-1}(i_r - 1).$$

The values $i_1, \dots, i_r$ corresponding to a given value of l are found by the following algorithm:

Operation 1, λ:

$$\frac{u_\lambda - u_{\lambda-1}}{n_1 \cdots n_{\lambda-1}} = d_\lambda \to \text{ if } u_{\lambda-1} = \begin{cases} 0, & \text{then } i_\lambda = d_\lambda,\ i_1 = n_1, \dots,\ i_{\lambda-1} = n_{\lambda-1}, \\ > 0, & \text{then } i_\lambda = d_\lambda + 1, \\ & \text{go to operation } 2, \lambda. \end{cases}$$

Operation 2, λ:

compare λ and 2 $\to$ if

$$\lambda = \begin{cases} 2, & \text{then } i_1 = u_1,\ \text{all } i_k \text{ have been found,} \\ > 2, & \text{then go to operation } 1,\ \lambda - 1. \end{cases}$$

$$\lambda = r, \dots, 1, \qquad \text{with } u_r = l.$$

Theorem 5. *If the intersection of all M_j is a nonempty set, then problem (1) –(2) reduces to another problem whose number of indices is*

$$\left| M \setminus \bigcap_{j=1}^{t} M_j \right|.$$

Proof. Let

$$\bigcap_{j=1}^{t} M_j = \{s - r + 1,\ s - r + 2, \dots, s\} = \overline{M}.$$

Then conditions (1) and (2) can be written in a somewhat different form:

$$\sum_{k \in M_j \setminus \overline{M}} \left(\sum_{k \in \overline{M}} x_{i_1 \dots i_s} \right) = a_{j(i_1 \dots i_s)}; \tag{1'}$$

$$L = \sum_{k \in M \setminus \overline{M}} \left(\sum_{k \in \overline{M}} p_{i_1 \dots i_s} x_{i_1 \dots i_s} \right). \tag{2'}$$

Denote

$$\sum_{k \in \overline{M}} x_{i_1 \dots i_s} = y_{i_1 \dots i_{s-r}}; \tag{4}$$

$$q_{i_1 \dots i_{s-r}} = \min_{1 \le i_k \le n_k,\ s-r+1 \le k \le s} p_{i_1 \dots i_s}. \tag{5}$$

Fig. 1

Figure 1: Fig. 1

Then from the optimality principle it is not difficult to understand that problem (1′)–(2′) reduces to the following:

Find the minimum of the functional

$$L = \sum_{k\in M\setminus \bar{M}} q_{i_1\dots i_{s-r}} y_{i_1\dots i_{s-r}}, \quad y_{i_1\dots i_{s-r}} \geq 0 \tag{6}$$

under the condition that

$$\sum_{k\in M_i\setminus \bar{M}} y_{i_1,\dots,i_{s-r}} = a_{j(i_1,\dots,i_s)}. \tag{7}$$

Let $y^{(0)}_{i_1\dots i_{s-r}}$ satisfy all conditions (7) and deliver the minimum of (6). Then all $x^{(0)}_{i_1\dots i_s}$ whose coordinates coincide with the coordinates

$$\min_{1\leq i_k\leq n_k,\ s-r+1\leq k\leq s} p_{i_1\dots i_s},$$

are equal to $y^{(0)}_{i_1\dots i_{s-r}}$, and the remaining $x^{(0)}_{i_1\dots i_s}$ are equal to zero. Thus, in the last case it is possible not only to reduce the problem to a simpler one, but also to reduce the number of unknowns.

Definition. Problems that do not satisfy the conditions of Theorems 2–5 will be called **irreducible**. All symmetric problems of type T_m are irreducible.

For $s = 4$ there are 8 irreducible nonsymmetric problems. Schematically, each of these problems can be represented as follows: let each index be denoted by a point; arrows between indices denote summation over those indices that lie at the arrowheads and inside them; a small circle around a point means that summation is carried out over a single index. All schemes were constructed symmetrically (see Fig. 1).

Fig. 1

Let $x^{(0)}_{i_1\dots i_s}$ be an optimal solution of the problem with conditions (1), (2).

Theorem 6. *Multiply all $a_{j(i_1,\dots,i_s)}$ by one and the same number $c > 0$. Then the new optimal solution has the form $cx^{(0)}_{i_1\dots i_s}$.*

Proof. The validity of the assertion is easy to understand if one takes into account that multiplying all $a_{j(i_1,\dots,i_s)}$ by c is equivalent to choosing a new unit of measurement, which should not affect the optimum.

Theorem 7. *Multiply all $p_{i_1 \dots i_s}$ by one and the same number $c > 0$. Then the optimal solution of the new problem will be the same as that of the original one, while the value of the linear form will increase by a factor of c.*

Proof is analogous to Theorem 6.

Theorem 8. *Consider a new problem in which all $p_{i_1 \dots i_s}$ corresponding to $x_{i_1 \dots i_s}$ that enter into some condition are either increased or decreased by one and the same number c. Then the optimal solution of the new problem will be the same as that of the original one, while the value of the linear form will increase or decrease by a constant quantity.*

Proof. Let $M_1 = \{1, 2, \dots, r\}$. Add to all $p_{i_1 \dots i_r 1 \dots 1}$ corresponding to the condition

$$\sum_{k \in M_1} x_{i_1 \dots i_r 1 \dots 1} = a_{1(1,\dots,1)}$$

a constant

number c. Then the functional takes the form

$$L_1 = \sum_{k \in M} p_{i_1 \dots i_s} x_{i_1 \dots i_s} - \sum_{k \in M_1} p_{i_1 \dots i_r 1 \dots 1} x_{i_1 \dots i_r 1 \dots 1} +$$

$$+ \sum_{k \in M_1} \left(p_{i_1 \dots i_r 1 \dots 1} + c \right) x_{i_1 \dots i_r 1 \dots 1} = \sum_{k \in M} p_{i_1 \dots i_s} x_{i_1 \dots i_s} + c a_{1(1,\dots,1)}.$$

Thus, $L_1 = L + c a_{1(1,\dots,1)}$, which is what was required to prove.

The last three properties can be used essentially in finding optimal solutions.

Institute of Complex Transport Problems
of Gosplan of the USSR

Received
27 X 1962

REFERENCES

[1] E. Shell, Proc. 2-d Symposium in linear Programming, Washington, 1, 1955.
[2] Progress in the Operation Research, 1961.
[3] B. S. Verkhovskii, On one algorithm for solving a multi-index transportation problem on an electronic computer, Novosibirsk, 1962.

Note: Figure translations are in progress. See original paper for figures.