G. V. VIRABYAN

1963

SovietRxiv

Abstract

Full Text

G. V. VIRABYAN

ON THE RESOLVENT OF AN OPERATOR

(Presented by Academician S. L. Sobolev on 2 II 1963)

It is known that the construction of the resolvent of the operator $\Pi = -\Delta^{-1}\Box$ of the type of S. L. Sobolev (Δ^{-1} is the operator inverse to the Laplace operator $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, and $\Box$ is the two-dimensional wave operator $\Box = \partial^2/\partial x^2 - \partial^2/\partial y^2$) in the Sobolev-Hilbert space $\mathring{W}_2^{(1)}$ reduces to the solution of the following boundary-value problem:

$$(1+\lambda+i\tau)\frac{\partial^2 u}{\partial x^2} - (1-\lambda-i\tau)\frac{\partial^2 u}{\partial y^2} = f(x_1, y), \tag{1}$$

$$u\big|_\Gamma = 0, \tag{2}$$

where Γ is the boundary of the domain Ω under consideration.

Therefore, the construction of the resolvent of the operator Π is equivalent to the construction of the Green' s function for the boundary-value problem (1), (2). As R. A. Aleksandryan showed ([1]), the fundamental solution of problem (1), (2) has the form

$$\gamma(P, P_0; \lambda, \tau) = \frac{1}{c(\lambda, \tau)} \ln\{[(\lambda-1)(x-x_0)^2 + (\lambda+1)(y-y_0)^2] +$$

$$+ i\tau[(x-x_0)^2 + (y-y_0)^2]\}; \tag{3}$$

$c(\lambda, \tau)$ is a normalizing factor; P is a point with coordinates x, y, and P_0 is a point with coordinates x_0, y_0.

Using the explicit expression for the fundamental solution (3), it is easy to construct ([1]) the Green' s function for problem (1), (2) in the case of a half-plane, in the form of the difference of the fundamental solutions at the points $P_0(x_0, y_0)$ and $P_1(x_0, -y_0)$:

$$G^{(0)}(P, P_0; \lambda, \tau) = \gamma(P, P_0; \lambda, \tau) - \gamma(P, P_1; \lambda, \tau) = \sum_{k=0}^{1} (-1)^k \gamma(P, P_k; \lambda, \tau).$$

In the present note we construct the Green' s function for the boundary-value problem (1), (2), and hence the resolvent of the operator Π, in the cases when the domain Ω is a quadrant of the plane, a strip, a half-strip, or a square.

The method of constructing the Green' s function in all the cases considered is as follows: an arbitrary point P_0 inside the domain Ω is taken, and all possible mirror images $\{P_k\}$ of this point with respect to the boundary Γ are considered. Then, by adding the fundamental solutions (3) at these points with the corresponding signs, one obtains the Green' s function for the domain Ω.

1°. Let the domain Ω be a quadrant of the plane ($x \geq 0,\ y \geq 0$), and let $P_0(x_0, y_0)$ be any point inside this quadrant. The possible mirror images of the point P_0 with respect to the boundaries of the domain under consideration, i.e. with respect to the straight lines $x = 0$ and $y = 0$, will be the points $P_1(-x_0, y_0)$, $P_2(-x_0, -y_0)$, $P_3(x_0, -y_0)$. Form the sum

$$G^{(1)}(P, P_0; \lambda, \tau) = \gamma(P, P_0; \lambda, \tau) - \gamma(P, P_1; \lambda, \tau) + \gamma(P, P_2; \lambda, \tau) -$$

$$-\gamma(P, P_3; \lambda, \tau) = \sum_{k=0}^{3} (-1)^k \gamma(P, P_k; \lambda, \tau). \tag{4}$$

Substituting the explicit expression for the fundamental solution (3) into (4), we ob-

we obtain

$$G(P, P_0; \lambda, \tau) = \frac{1}{c(\lambda, \tau)} \ln\Big\{\big[(\lambda - 1)(x - x_0)^2 + (\lambda + 1)(y - y_0)^2\big] +$$

$$+ i\tau\big[(x - x_0)^2 + (y - y_0)^2\big]\Big\}\Big\{\big[(\lambda - 1)(x + x_0)^2 + (\lambda + 1)(y + y_0)^2\big] +$$

$$+ i\tau\big[(x + x_0)^2 + (y + y_0)^2\big]\Big\}\Big\{\big[(\lambda - 1)(x + x_0)^2 + (\lambda + 1)(y - y_0)^2\big] +$$

$$+ i\tau\big[(x + x_0)^2 + (y - y_0)^2\big]\Big\}^{-1}\Big\{\big[(\lambda - 1)(x - x_0)^2 + (\lambda + 1)(y + y_0)^2\big] +$$

$$+ i\tau\big[(x - x_0)^2 + (y + y_0)^2\big]\Big\}^{-1}.$$

By direct verification it is easy to see that the function obtained is the Green' s function for the boundary-value problem (1), (2) in the case of a quadrant.

2°. Let the domain Ω be the strip $(-\infty < x < +\infty,\ 0 \leq y \leq 1)$. Unlike the case considered above, here every interior point $P_0(x_0, y_0)$ of the strip will have an infinite number of mirror images with respect to the boundaries $y = 0$, $y = 1$ of the strip. All possible such images of the point P_0 will have coordinates

$$P_k^{(1)}(x_0, 2k + y_0), \quad P_k^{(2)}(x_0, -2k + y_0), \quad P_k^{(3)}(x_0, 2k - y_0), \quad P_k^{(4)}(x_0, -2k - y_0),$$

$$k = 0, 1, 2, \dots .$$

Let us form the following series of fundamental solutions:

$$G^{(2)}(P, P_0; \lambda, \tau) = G^{(0)}(P, P_0; \lambda, \tau) + \sum_{k=1}^{\infty} \{\gamma(P, P_k^{(1)}; \lambda, \tau) + \gamma(P, P_k^{(2)}; \lambda, \tau) -$$

$$-\gamma(P, P_k^{(3)}; \lambda, \tau) - \gamma(P, P_k^{(4)}; \lambda, \tau)\}. \tag{5}$$

It can be proved that the series (5) converges uniformly and absolutely in every finite part of the plane. Moreover, it proves possible to find the sum of this series in closed form. Indeed, substituting the value of the fundamental solution from (3) into (5), we obtain

$$G^{(2)}(P, P_0; \lambda, \tau) = \frac{1}{c(\lambda, \tau)} \ln\Big\{[(\lambda - 1)(x - x_0)^2 + (\lambda + 1)(y - y_0)^2] +$$

$$+ i\tau[(x - x_0)^2 + (y - y_0)^2]\Big\}\Big\{[(\lambda - 1)(x - x_0)^2 + (\lambda + 1)(y + y_0)^2] +$$

$$+ i\tau[(x - x_0)^2 + (y + y_0)^2]\Big\}^{-1} + \frac{1}{c(\lambda, \tau)} \sum_{k=1}^{\infty} \ln\Big\{[(\lambda - 1)(x - x_0)^2 +$$

$$+ (\lambda + 1)(y - y_0 - 2k)^2] + i\tau[(x - x_0)^2 + (y - y_0 - 2k)^2]\Big\} \times$$

$$\times \Big\{[(\lambda - 1)(x - x_0)^2 + (\lambda + 1)(y - y_0 + 2k)^2] + i\tau[(x - x_0)^2 +$$

$$+ (y - y_0 + 2k)^2]\Big\}\Big\{[(\lambda - 1)(x - x_0)^2 + (\lambda + 1)(y + y_0 - 2k)^2] +$$

$$+ i\tau[(x - x_0)^2 + (y + y_0 - 2k)^2]\Big\}^{-1}\Big\{[(\lambda - 1)(x - x_0)^2 + (\lambda + 1) \times$$

$$\times(y+y_0+2k)^2\big]+i\tau\big[(x-x_0)^2+(y+y_0+2k)^2\big]\Big\}^{-1}. \qquad (6)$$

Expanding in the resulting series the expressions under the logarithm into factors, passing from the sum to a product, and using the Weierstrass form of the gamma function (2), we find the sum of the series (6) in closed form through the gamma function:

$$G^{(2)}(P,P_0;\lambda,\tau)=\frac{1}{c(\lambda,\tau)}\ln\Big\{\big[(\lambda-1)(x-x_0)^2+(\lambda+1)(y-y_0)^2\big]+$$

$$+i\tau\big[(x-x_0)^2+(y-y_0)^2\big]\Big\}\Big\{\big[(\lambda-1)(x-x_0)^2+(\lambda+1)(y+y_0)^2\big]+$$

$$+i\tau\big[(x-x_0)^2+(y+y_0)^2\big]\Big\}^{-1}+$$

$$+\frac{1}{c(\lambda,\tau)}\ln\Bigg\{\Big\{\Gamma\Big[1-\frac{y+y_0}{2}-\nu\frac{(x-x_0)}{2}\Big]\Gamma\Big[1-\frac{y+y_0}{2}+\nu\frac{(x-x_0)}{2}\Big]\times$$

$$\times\Gamma\Big[1+\frac{y+y_0}{2}-\nu\frac{(x-x_0)}{2}\Big]\Gamma\Big[1+\frac{y+y_0}{2}+\nu\frac{(x-x_0)}{2}\Big]\Big\}\times$$

$$\times\Big\{\Gamma\Big[1-\frac{y-y_0}{2}-\nu\frac{(x-x_0)}{2}\Big]\Gamma\Big[1-\frac{y-y_0}{2}+\nu\frac{(x-x_0)}{2}\Big]\times$$

$$\times\Gamma\Big[1+\frac{y-y_0}{2}-\nu\frac{(x-x_0)}{2}\Big]\Gamma\Big[1+\frac{y-y_0}{2}+\nu\frac{(x-x_0)}{2}\Big]\Big\}^{-1}\Bigg\},$$

where

$$\nu=\sqrt{(1-\lambda-i\tau)/(1+\lambda+i\tau)}.$$

Next, using the known properties of the gamma function and passing from the gamma function to trigonometric functions by the formula

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z},$$

for the sum of series (6) we finally obtain

$$G^{(2)}(P,P_0;\lambda,\tau)=\frac{1}{c(\lambda,\tau)}\ln\{\{\sin\tfrac{1}{2}\pi[(y-y_0)+\nu(x-x_0)]\times \times\sin\tfrac{1}{2}\pi[(y-y_0)-\nu(x-x_0)]\}\{\sin\tfrac{1}{2}\pi[(y+y_0)+\nu(x-x_0)]\times \times\sin\tfrac{1}{2}\pi[(y+y_0)-\nu(x-x_0)]\}^{-1}\}. \tag{7}$$

By direct verification one can ascertain that the function (7) represents the Green' s function for the boundary-value problem (1), (2) in the case of a strip.

3°. Let the domain Ω be a half-strip $(0\le x<+\infty,\ 0\le y\le 1)$. In this case the Green' s function can be constructed with the aid of the Green' s function for the strip. Indeed, the function

$$\begin{aligned}G^{(3)}(P,P_0;\lambda,\tau)&=G^{(2)}(P,P_0'(x_0,y_0);\lambda,\tau)-G^{(2)}(P,P_1(-x_0,y_0);\lambda,\tau)\\&=\frac{1}{c(\lambda,\tau)}\ln\{\{\sin\frac{\pi}{2}[(y-y_0)+\nu(x-x_0)]\sin\frac{\pi}{2}[(y-y_0)-\nu(x-x_0)]\times\\&\times\sin\frac{\pi}{2}[(y+y_0)+\nu(x+x_0)]\sin\frac{\pi}{2}[(y+y_0)-\nu(x+x_0)]\}\times\\&\times\{\sin\frac{\pi}{2}[(y+y_0)+\nu(x-x_0)]\sin\frac{\pi}{2}[(y+y_0)-\nu(x-x_0)]\times\\&\times\sin\frac{\pi}{2}[(y-y_0)+\nu(x+x_0)]\sin\frac{\pi}{2}[(y-y_0)-\nu(x+x_0)]\}^{-1}\}\end{aligned}$$

represents the Green' s function for problem (1), (2) in the case of a half-strip.

4°. Let the domain Ω be a square $(0\le x\le 1,\ 0\le y\le 1)$. Take an arbitrary interior point $P_0(x_0,y_0)$ of this square and consider all possible mirror images of the point P_0 with respect to the sides $x=0$, $x=1$. The points obtained will have coordinates $P_l^{(1)}(2l+x_0,y_0)$, $P_l^{(2)}(-2l+x_0,y_0)$, $P_l^{(3)}(2l-x_0,y_0)$, $P_l^{(4)}(-2l-x_0,y_0)$, $l=0,1,2,\ldots.$ Form the following series from the Green' s functions for the strip (7):

$$\begin{aligned}G^{(4)}(P,P_0;\lambda,\tau)&=G^{(3)}(P,P_0;\lambda,\tau)+\sum_{l=1}^{\infty}\{G^{(2)}(P,P_l^{(1)};\lambda,\tau)+\\&+G^{(2)}(P,P_l^{(2)};\lambda,\tau)-G^{(2)}(P,P_l^{(3)};\lambda,\tau)-G^{(2)}(P,P_l^{(4)};\lambda,\tau)\}.\end{aligned}\tag{8}$$

Substituting the expression for the Green' s function for the strip (7) into (8), we obtain

$$G^{(4)}(P,P_0;\lambda,\tau)=\frac{1}{c(\lambda,\tau)}\ln\{\{\sin\frac{\pi}{2}[(y-y_0)+\nu(x-x_0)]\times$$
$$\times\sin\frac{\pi}{2}[(y-y_0)-\nu(x-x_0)]\sin\frac{\pi}{2}[(y+y_0)+\nu(x+x_0)]\times$$
$$\times\sin\frac{\pi}{2}[(y+y_0)-\nu(x+x_0)]\}\{\sin\frac{\pi}{2}[(y+y_0)+\nu(x-x_0)]\times$$
$$\times\sin\frac{\pi}{2}[(y+y_0)-\nu(x-x_0)]\sin\frac{\pi}{2}[(y-y_0)+\nu(x+x_0)]\times$$
$$\times\sin\frac{\pi}{2}[(y-y_0)-\nu(x+x_0)]\}^{-1}\}$$
$$+\frac{1}{c(\lambda,\tau)}\sum_{l=1}^{\infty}\ln\{\{\sin\frac{\pi}{2}[(y-y_0)+$$
$$+\nu(x-x_0-2l)]\sin\frac{\pi}{2}[(y-y_0)-\nu(x-x_0-2l)]\sin\frac{\pi}{2}[(y-y_0)+$$
$$+\nu(x-x_0+2l)]\sin\frac{\pi}{2}[(y-y_0)-\nu(x-x_0+2l)]\}\{\sin\frac{\pi}{2}[(y+y_0)+$$
$$+\nu(x-x_0-2l)]\sin\frac{\pi}{2}[(y+y_0)-\nu(x-x_0-2l)]\sin\frac{\pi}{2}[(y+y_0)+$$
$$+\nu(x-x_0+2l)]\sin\frac{\pi}{2}[(y+y_0)-\nu(x-x_0+2l)]\}^{-1}\{\sin\frac{\pi}{2}[(y+y_0)+$$

$$\begin{gathered}+v(x+x_0-2l)\sin\frac{\pi}{2}\left[(y+y_0)-v(x+x_0-2l)\right]\sin\frac{\pi}{2}\left[(y+y_0)+\right.\\ +v(x+x_0+2l)\sin\frac{\pi}{2}\left[(y+y_0)-v(x+x_0+2l)\right]\Big\}\Big\{\sin\frac{\pi}{2}\left[(y-y_0)+\right.\\ +v(x+x_0-2l)]\sin\frac{\pi}{2}\left[(y-y_0)-v(x+x_0-2l)\right]\sin\frac{\pi}{2}\left[(y-y_0)+\right.\\ +v(x+x_0+2l)]\sin\frac{\pi}{2}\left[(y-y_0)-v(x+x_0+2l)\right]\Big\}^{-1}\Big\}.\end{gathered}\tag{9}$$

It can be proved that the series obtained converges uniformly and absolutely and that the sum of this series represents the Green function for problem (1), (2) in the case of the square under consideration. Moreover, it turns out to be possible to sum this series. Indeed, differentiating series (9) with respect to x and y, using the known representation of the Weierstrass ζ-function [3]

$$\zeta(u)=\frac{\zeta(\omega_1)}{\omega_1}u+\frac{\pi}{2\omega_1}\operatorname{ctg}\frac{\pi u}{2\omega_1}+$$

$$+\frac{\pi}{2\omega_1}\sum_{m=1}^{\infty}\left\{\operatorname{ctg}\left(\frac{\pi u}{2\omega_1}+m\pi\frac{\omega_2}{\omega_1}\right)+\operatorname{ctg}\left(\frac{\pi u}{2\omega_1}-m\pi\frac{\omega_2}{\omega_1}\right)\right\}$$

and passing from the ζ-function to the function σ, we finally obtain the Green function for the square in finite form through the doubly periodic Weierstrass function:

$$G^{(4)}(P, P_0; \lambda, \tau) = \frac{1}{c(\lambda, \tau)} \ln \Big\{ \{\sigma[(y - y_0) + v(x - x_0)]\sigma[(y - y_0) - \\ - v(x - x_0)]\sigma[(y + y_0) + v(x + x_0)]\sigma[(y + y_0) - v(x + x_0)]\} \times \\ \times \{\sigma[(y + y_0) + v(x - x_0)]\sigma[(y + y_0) - v(x - x_0)]\sigma(y - y_0) + \\ + v(x + x_0)]\sigma[(y - y_0) - v(x + x_0)]\}^{-1} \Big\}.$$

Let us note that in all the cases considered the Green function for the boundary-value problem, and consequently also the resolvent of the operator Π, has been constructed with the aid of only one fundamental solution of problem (1), (2).

5°. In exactly the same way as above one can construct Green functions in all the cases considered above for the problem adjoint to (1), (2),

$$(1 + \lambda - i\tau)\frac{\partial^2 u}{\partial x^2} - (1 - \lambda + i\tau)\frac{\partial^2 u}{\partial y^2} = \Phi(x, y), \tag{1*}$$

$$u\big|_{\Gamma} = 0. \tag{2*}$$

Using the finite representation of the Green function for the square, forming the difference of the Green functions of problems (1), (2) and (1*), (2*) and passing to the limit as $\tau \to +0$, we obtain the proper function of the operator Π or else zero, depending on whether the parameter

$$\mu = \sqrt{(1 - \lambda)/(1 + \lambda)}$$

is a rational or an irrational number. This fact is proved with the aid of a result of Jacobi [4], according to which a doubly periodic function reduces to a constant if the ratio of the periods is irrational, or else to a simply periodic function if the ratio of the periods is rational.

I take this opportunity to express my sincere gratitude to my teachers S. L. Sobolev and R. A. Aleksandryan for the formulation of the problem and for useful advice in carrying out the present work.

Computing Center
Academy of Sciences of the Armenian SSR

Received
4 XI 1962

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Note: Figure translations are in progress. See original paper for figures.