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Abstract

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DIFFERENTIAL PROPERTIES OF SOLUTIONS OF CERTAIN HIGHER-ORDER DIFFERENTIAL EQUATIONS

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In this work we investigate the differential properties of solutions of certain boundary-value problems depending on the differential properties of the boundary functions in terms of the H-classes and W-classes of S. L. Sobolev and generalized classes.

We consider the Dirichlet problem for the equation

$$L^{(l,s)}u \equiv (-1)^{l+1}\frac{\partial^{2l}u}{\partial x^{2l}} + (-1)^{s+1}\frac{\partial^{2s}u}{\partial y^{2s}} = 0 \tag{1}$$

(l, s are natural numbers, $y > 0$, $-\infty < x < \infty$) under the conditions

$$\left.\frac{\partial^j u}{\partial y^j}\right|_{y=0} = \varphi_j(x) \qquad (j = 1, 2, \dots, s-1) \tag{2}$$

and the function $u(x, y)$ is bounded in $R_2^0 = \{y > 0;\ -\infty < x < \infty\}$.

The conditions (2) are understood in the sense that

$$\lim_{y\to+0}\int_{-\infty}^{\infty}\left|\frac{\partial^j u(x,y)}{\partial y^j} - \varphi_j(x)\right|^p dx = 0 \qquad (1 < p < \infty). \tag{3}$$

Let us note that for $l = s = 1$, $L^{(1,1)}u \equiv \Delta u$, where Δ is the Laplace operator. Applying the Fourier method of separation of variables, we obtained an analytic representation of the solution of the boundary-value problem (1)–(2). In particular, for $s = 1$ the solution of the boundary-value problem has the form

$$u(x, y) = \int_{-\infty}^{\infty} K_0(t, y; l, 1)\varphi_0(x + t)\, dt, \tag{4}$$

where

$$K_0(t,y;l,1)=\frac{1}{2\pi}\int_0^\infty e^{-\lambda^l y}\cos\lambda t\,d\lambda;$$

for $s=2$

$$u(x,y)=\int_{-\infty}^\infty K_0(t,y;l,2)\varphi_0(x+t)\,dt+\int_{-\infty}^\infty K_1(t,y;l,2)\varphi_1(x+t)\,dt, \qquad (5)$$

where

$$K_0(t,y;l,2)=\frac{1}{2\pi}\int_0^\infty e^{-\nu y}(\cos\nu y+\sin\nu y)\cos\lambda t\,d\lambda,$$

$$K_1(t,y;l,2)=\frac{1}{2\pi}\int_0^\infty e^{-\nu y}\frac{\sin\nu y}{\nu}\cos\lambda t\,d\lambda \qquad \left(\nu=\frac{\sqrt{2}}{2}\lambda^{l/2}\right);$$

for $s=3$

$$u(x,y)=\int_{-\infty}^\infty K_0(t,y;l,3)\varphi_0(x+t)\,dt+\int_{-\infty}^\infty K_1(t,y;l,3)\varphi_1(x+t)\,dt+$$

$$+\int_{-\infty}^\infty K_2(t,y;l,3)\varphi_2(x+t)\,dt,$$

where

$$K_0(t,y;l,3)=\frac{1}{2\pi}\int_0^\infty\left[e^{-y\lambda^\alpha}+\frac{2\sqrt{3}}{3}e^{-\frac{y}{2}\lambda^\alpha}\sin\frac{\sqrt{3}}{2}y\lambda^\alpha\right]\cos\lambda t\,d\lambda \quad \left(\alpha=\frac{l}{3}\right),$$

$$K_1(t,y;l,3)=$$

$$=\frac{1}{2\pi}\int_{-\infty}^\infty\lambda^{-\alpha}\left[e^{-y\lambda^\alpha}+e^{-\frac{y}{2}\lambda^\alpha}\left(\sqrt{3}\sin\frac{\sqrt{3}}{2}y\lambda^\alpha-\cos\frac{\sqrt{3}}{2}y\lambda^\alpha\right)\right]\cos\lambda t\,d\lambda,$$

$$K_2(t,y;l,3)=$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \lambda^{-2\lambda}\left[e^{-y\lambda^{\alpha}} + e^{-\frac{y}{2}\lambda^{\alpha}}\left(\frac{\sqrt{3}}{3}\sin\frac{\sqrt{3}}{2}y\lambda^{\alpha} - \cos\frac{\sqrt{3}}{2}y\lambda^{\alpha}\right)\right]\cos\lambda t\, d\lambda.$$

Using the explicit representation of the solution of the boundary-value problem, on the basis of the properties of the kernels $K_j(t, y; , l, s)$ $(j = 0, 1, \dots, s-1)$, which we call generalized Poisson kernels, and with the aid of the Hardy, Hölder, and Minkowski inequalities, we proved a number of theorems generalizing the corresponding results of S. M. Nikol' skii ([1]), O. V. Besov ([3]), S. V. Uspenskii ([5,6]), L. D. Kudryavtsev ([7]), and others.

Let $R_1 = \{-\infty < x < \infty\}$, $R_2(0,1) = \{0 < y < 1,\ -\infty < x < \infty\}$.

Theorem 1. If the functions $\varphi_j(x) \in H_p^{\left[r+(s-1-j)\frac{l}{s}\right]}(R_1)$ $(r > 0;\ 1 < p < \infty;\ j = 0, 1, \dots, s-1)$, then the solution of the boundary-value problem (1)–(2)

$$u(x,y) \in H_{px}^{\left[r+\left(s-1+\frac{1}{p}\right)\frac{l}{s}\right]}[R_2(0,1)]$$

for any $r > 0$;

$$u(x,y) \in H_{py}^{\left(\frac{sr}{l}+s-1+\frac{1}{p}\right)}[R_2(0,1)]$$

for $p = 2$, r arbitrary; for $p \neq 2$, $r + (s-1-j)\frac{l}{s} \neq 2k$ ($2k$ an even number)*.

Theorem 2. If the functions $\varphi_j(x) \in W_p^{\left[r+(s-1-j)\frac{l}{s}\right]}(R_1)$ $(1 < p < \infty;\ j = 0, 1, \dots, s-1;\ r + (s-1-j)\frac{l}{s}$ are not integers for $p \neq 2)$, then the solution of problem (1)–(2)

$$u(x,y) \in W_p^{(r_1,r_2)}[R_2(0,1)],$$

where

$$r_1 = r + \left(s - 1 + \frac{1}{p}\right)\frac{l}{s}, \qquad r_2 = \frac{s}{l}r_1$$

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* For the definition of the classes $H_p^{(r_1,r_2)}$, see ([2]).

** For the definition of the classes $W_p^{(r_1,r_2)}$, see ([8,4,5]).

The proof of Theorems 1 and 2 was carried out by us for $s = 1, 2, 3$ and arbitrary natural l. These theorems cannot be improved in these terms, as follows from the results of S. M. Nikol' skii ([2]), p. 296, and O. V. Besov ([4]), p. 79.

Further, we have shown that the solution of problem (1)–(2) has derivatives of arbitrary order, summable with a weight. For example, for $s = 1$ the following holds.

Theorem 3. Let $\varphi_0(x) \in W_p^{(r)}(R_1)$ ($r > 0$; $r = \rho + d$, $0 < \alpha < 1$; ρ an integer); $\frac{r}{l} + \frac{1}{p} = n + \beta$, $0 \leqslant \beta < 1$; $r + \frac{l}{p} = nl + l\beta = nl + k + \gamma$, $k = 0, 1, \dots, l-1$, $0 \leqslant \gamma < 1$, n an integer.

Then the function (4) has derivatives of order higher than n with respect to y and of order higher than $nl + k$ with respect to x, summable with a weight:

$$\left\| \frac{\partial^{n+s_1+s_2}}{\partial y^{n+s_1} \partial x^{s_2}} \, y^{s_1 - \beta + \frac{s_2}{l}} \right\|_{L_p[R_2(0,1)]} \leqslant c \|\varphi_0\|_{W_p^{(r)}(R_1)},$$

where $s_1 = 1, 2, \dots;$ $s_2 = 0, 1, \dots;$

$$\left\| \frac{\partial^{nl+k+s_1+s_2} u}{\partial y^{s_1} \partial x^{nl+k+s_2}} \, y^{s_1 + \frac{s_2 - \gamma}{l}} \right\|_{L_p[R_2(0,1)]} \leqslant c \|\varphi_0\|_{W_p^{(r)}(R_1)},$$

where $s = 0, 1, \dots;$ $s_2 = 1, 2, \dots,$

$$\|\psi\|_{L_p[R_2(0,1)]} = \left(\int_0^1 \int_{-\infty}^{\infty} |\psi|^p \, dx \, dy \right)^{\frac{1}{p}}.$$

This result is definitive and generalizes the result of S. V. Uspenskii ([6]) (Theorem 2).

Using Theorems 1 and 2, one can obtain sufficient conditions for the boundedness of the generalized Dirichlet integral

$$D(u; l, s)_p = \int_0^{\infty} \int_{-\infty}^{\infty} \left[\left| \frac{\partial^l u}{\partial x^l} \right|^p + \left| \frac{\partial^s u}{\partial y^s} \right|^p \right] dx \, dy$$

in terms of H-classes and W-classes. We give these conditions for $s = 2$.

Theorem 4. Let the functions $\varphi_0 \in W_n^{\left(\frac{l}{2q} + \frac{l}{2}\right)}(R_1)$, $\varphi_1 \in W_n^{\left(\frac{l}{2q}\right)}(R_1)$, where $\frac{1}{p} + \frac{1}{q} = 1$ (for $p \neq 2$, $\frac{l}{2q}$, $\frac{l}{2q} + \frac{l}{2}$ are not integers).

Then these functions can be continued to R_2^0 in the form of the solution (5), with $D(u; l, 2)_p < \infty$, and conditions (2) are satisfied.

In terms of H-classes, a similar theorem is formulated with accuracy up to an arbitrary $\varepsilon > 0$.

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Note: Figure translations are in progress. See original paper for figures.