ON EQUIVALENT TRANSFORMATIONS OF LOGICAL NETWORKS

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Fig. 1

Figure 1: Fig. 1

Abstract

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CYBERNETICS AND CONTROL THEORY

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ON EQUIVALENT TRANSFORMATIONS OF LOGICAL NETWORKS

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This note considers equivalent transformations of logical networks by replacing their subnetworks with equivalent ones. For a whole series of classes of logical networks, N. A. Gorbunishchaya constructed finite systems of transformation rules that make it possible to transform any two equivalent networks of a given class into one another in such a way that all intermediate networks belong to the same class ([1]). Here the existence of a finite system of rules will be proved for classes of networks with a number of delay elements not exceeding a given number k ($k = 0, 1, 2, ...$), and the nonexistence of such a system for the class of all networks.

1. By a **logical network**, or simply a **network**, we shall mean any circuit having the form a (Fig. 1), where C is a circuit of functional elements ([2])* with $m+k$ inputs and $n+k$ outputs; the letter z denotes each of the k delay elements; $x_1, x_2, \dots, x_m$ are the inputs of the network; $y_1, y_2, \dots, y_n$ are its outputs. It is assumed here that C contains functional elements of only a finite number of admissible types and that the functions of the algebra of logic realized by elements of admissible types form a complete system**.

Fig. 1

If a sequence of states of the network inputs is given,

$$(x_1^t, \dots, x_n^t), \quad t = 0, 1, 2, \dots \quad (x_i^t = 0 \text{ or } 1),$$

then, in the usual way, the sequence of states of its outputs is determined (it is assumed here that the outputs of the delays at $t = 0$ are in state 0). Two networks A and B, with the same number of inputs and outputs, between whose inputs and between whose outputs a one-to-one correspondence has been established, are called **equivalent** (for the given correspondence between the inputs and outputs of A and B) if, for identical sequences at the corresponding inputs of A and B, the sequences at the corresponding outputs of A and B are also identical.

We shall call a **subnetwork** of a network A any network B whose elements belong to A, such that: 1) every input (output) of the network A that belongs to B is an input (output) of B; 2) every vertex of the network B that is an input of some element of the network A not belonging to B, and every out–

* The requirement from ([2]) that the state of the output of a functional element be a function depending essentially on the state of each input is not obligatory here.

** The concept of a network introduced here differs only inessentially (namely, in the method of realizing the circuit C) from the concept of a completely regular network in ([3]).

input of some element of the network B, is an output of B; 3) every vertex of the network B that is an input or output of some element of the network A, does not belong to B, and is not an output of any element of the network B, is an input of B. It is easy to verify that every network goes over into an equivalent one when a subnetwork is replaced by an equivalent subnetwork. Therefore each pair of equivalent networks A_1 and A_2 specifies a rule for an equivalent transformation of logical networks, according to which, in any network, a subnetwork isomorphic to A_1 may be replaced by A_2, and conversely (cf. (4)).

2. Let $u_1, u_2, \dots, u_s$ be some vertices of the network A. We shall denote by $\tilde{u}_t^A$ the set of states of these vertices at time t; the indices A and t will sometimes be omitted. We shall call an **operator** any mapping of an m-dimensional unit cube* into an n-dimensional unit cube (for some m and n); if m and n must be indicated explicitly, we shall speak of an (m, n)-operator. Let $\Phi(\tilde{x}, \tilde{z})$ be an $(m + k, s)$-operator in the arguments $x_1, \dots, x_m, z_1, \dots, z_k$. For each set of constants $\tilde{x} = (x_1, \dots, x_m)$ we agree to denote by $\Phi_{\tilde{x}}(\tilde{z})$, or simply $\Phi_{\tilde{x}}$, the (k, s)-operator obtained from Φ by fixing $\tilde{x}$. Analogously we introduce the notation $\Phi_{\tilde{z}}$.

Networks containing no delay elements are simply circuits made of functional elements; the equivalence of such networks consists in the coincidence of the operators realized by the corresponding circuits of functional elements. In (1) a finite system of transformation rules was constructed that makes it possible to obtain from one another any equivalent networks without delays, and, moreover, the networks in the rules of this system themselves contain no delays. We shall denote this system by S_0. We shall call two networks with the same number of inputs, outputs, and delays **almost identical** (under a definite correspondence between the inputs of the networks and between their outputs) if they have the form a (Fig. 1) with equivalent (under a certain "extension" of the given correspondence to the inputs and outputs of the delays) C. It is clear that almost identical networks are equivalent, that two networks are almost identical

Fig. 2

Figure 2: Fig. 2

if and only if they can be obtained from one another by the rules S_0, and that specifying the operator realized by the circuit C (Fig. 1a) determines the network uniquely up to "almost identity" (provided it is indicated which outputs of C are connected with which inputs of it by delay elements). Let A be a network with inputs $x_1, \dots, x_m$ and outputs $y_1, \dots, y_n$; denote the outputs of the delays by $z_1, \dots, z_k$. It is easy to see that there exists a network A', having the form b (Fig. 1), such that A and A' are almost identical. In this case the operators realized by the circuits A_1 and A_2 are determined by the network A uniquely; we shall denote the first of them by $F^A(\tilde{x}, \tilde{z})$, and the second by $G^A(\tilde{x}, \tilde{z})$.

3. For each k there exist exactly $2^{k\cdot 2^k}$ distinct (k,k)-operators. Fix (for each k) a definite one-to-one correspondence between (k,k)-operators and strings of zeros and ones of length $k\cdot 2^k$. The string corresponding to an operator Φ will be denoted by $K(\Phi)$; the operator corresponding to the string $\tilde{\xi} = (\xi_1, \dots, \xi_{k\cdot 2^k})$ will be denoted by $H_{\tilde{\xi}}$. Let A and B be networks with m inputs, n outputs, and k delay elements. Among the (m,n)-operators $G^A_{\tilde{z}}(\tilde{x})$, $G^B_{\tilde{z}}(\tilde{x})$, for all possible $\tilde{z}$, there are no more than 2^{k+1} distinct ones; denote them by $\Omega_1(\tilde{x}), \Omega_2(\tilde{x}), \dots, \Omega_l(\tilde{x})$. Fix l distinct strings $\sigma_1, \dots, \sigma_l$ ($l \le 2^{k+1}$) of length $k+1$. Construct the network P, shown in Fig. 2, in which $B_1, B_2, B_3, B_4, B_5, B_6$ are circuits of functional elements realizing respectively the following operators $\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5, \Phi_6$: $\Phi_1(\tilde{x}) = K(F^A_{\tilde{x}})$; $\Phi_2(\tilde{x}) = K(F^B_{\tilde{x}})$; $\Phi_3(\tilde{u}, \tilde{v}) = \tilde{u}$, if there exists $\tilde{x}_0$ such that $\Phi_1(\tilde{x}_0) = \tilde{u}$ and $\Phi_2(\tilde{x}_0) = \tilde{v}$, and otherwise $\Phi_3(\tilde{u}, \tilde{v}) = K(F^A_{\tilde{0}}(\tilde{z}))$, where $\tilde{0} = (0, 0, \dots, 0)$; $\Phi_4(\tilde{z}, \tilde{\xi}) = H_{\tilde{\xi}}(\tilde{z})$; $\Phi_5(\tilde{z}) = \sigma_i$,

* That is, the set of all strings of length m consisting of zeros and ones.

if $G^A_{\tilde{z}}$ is Ω_i; $\Phi_6(\tilde{x}, \tilde{w}) = \Omega_i(\tilde{x})$, if for some (and hence unique) i $\tilde{w} = \sigma_i$; otherwise $\Phi_6(\tilde{x}, \tilde{w}) = \tilde{0}$. Next construct a network Q, obtained from P by replacing B_3 by B'_3 and B_5 by B'_5; B'_3 and B'_5 realize the following operators: $\Phi'_3(\tilde{u}, \tilde{v}) = \tilde{v}$, if for some $\tilde{x}_0$ $\Phi_1(\tilde{x}_0) = \tilde{u}$ and $\Phi_2(\tilde{x}_0) = \tilde{v}$; otherwise $\Phi'_3(\tilde{u}, \tilde{v}) = K(F^B_0(\tilde{z}))$; $\Phi'_5(\tilde{z}) = \sigma_i$, if $G^B_{\tilde{z}}$ is Ω_i. It is verified directly that A and P are almost identical and that B and Q are almost identical*.

Fig. 2

Let now P' be the subnetwork of the network P consisting of all delay elements and the circuits B_3, B_4, B_5 with inputs $\tilde{u}$ and $\tilde{v}$ and outputs $\tilde{w}$; let Q' be the corresponding subnetwork of the network Q.

Lemma 1. *If A and B are equivalent, then P' and Q' are equivalent.*

Proof. Since P is equivalent to A, and Q is equivalent to B, it is enough to prove that if P' and Q' are not equivalent, then P and Q are not equivalent.

Let the sets α_t, β_t of length $k \cdot 2^k$ $(t = 0, 1, \dots, T-1)$ be such that, if $\tilde{u}_t^{P'} = \tilde{u}_t^{Q'} = \alpha_t$ and $\tilde{v}_t^{P'} = \tilde{v}_t^{Q'} = \beta_t$ for $t = 0, 1, \dots, T-1$, then $\tilde{w}_T^{P'} \neq \tilde{w}_T^{Q'}$**. Let $\tilde{w}_T^{P'} = \sigma_{i_1}$, $\tilde{w}_T^{Q'} = \sigma_{i_2}$, $i_1 \neq i_2$; let the set $\tilde{\eta}$ be such that

$$\Omega_{i_1}(\tilde{\eta}) \neq \Omega_{i_2}(\tilde{\eta}). \tag{1}$$

Put, for $t = 0, 1, \dots, T-1$, $\tilde{\xi}_t = \tilde{0}$ (m zeros), if there does not exist a $\tilde{\xi}$ such that $\alpha_t = \Phi_1(\tilde{\xi})$ and $\beta_t = \Phi_2(\tilde{\xi})$; if such a $\tilde{\xi} = \tilde{\xi}^0$ exists, then put $\tilde{\xi}_t = \tilde{\xi}^0$.

We shall prove that if $\tilde{x}_t^P = \tilde{x}_t^Q = \tilde{\xi}_t$ $(t = 0, 1, \dots, T-1)$ and $\tilde{x}_T^P = \tilde{x}_T^Q = \tilde{\eta}$, then $\tilde{y}_T^P \neq \tilde{y}_T^Q$; then the lemma will be proved. In view of (1) and the definition of Φ_6, it is enough to show that $\tilde{w}_T^P = \tilde{w}_T^{P'}$ and $\tilde{w}_T^Q = \tilde{w}_T^{Q'}$ for the values under consideration of $\tilde{x}_t$ for P and Q, and $\tilde{u}_t$ and $\tilde{v}_t$ for P' and Q'. By the definition of the sequence $\{\tilde{\xi}_t\}$ and the definitions of Φ_1, Φ_2, Φ_3, we have $\tilde{\zeta}_t^P = \tilde{\zeta}_t^{P'}$ $(t = 0, 1, \dots, T-1)$, therefore $\tilde{z}_t^P = \tilde{z}_t^{P'}$ $(t = 0, 1, \dots, T)$; in particular, $\tilde{z}_T^P = \tilde{z}_T^{P'}$, whence $\tilde{w}_T^P = \tilde{w}_T^{P'}$. For Q and Q' the proof is analogous.

Theorem 1. *For every $k \geq 0$ there exists a finite system of rules allowing one to transform into one another any two equivalent networks with no more than k delay elements in such a way that all intermediate networks also contain no more than k delay elements.*

Proof. For the case $k = 0$ the theorem was proved in (1); the corresponding system of rules was denoted by S_0. Let $k > 0$. There exists a rule allowing one, in any network, to increase the number of delays by 1. Therefore, for the proof of the theorem it is enough to construct a finite system of rules allowing one to transform into one another any two equivalent networks with exactly k delays.

* The operator Φ_3 was defined "in parts" : by different formulas on two sets of input sets. The almost identity of A and P is connected only with the first part of the definition; "making the definition precise," necessary in order that Lemma 1 (see below) hold, may be carried out in various ways; the same applies to Φ_3'.

** In the networks P' and Q', $\tilde{w}_T$ is uniquely determined by $\tilde{u}_0, \dots, \tilde{u}_{T-1}, \tilde{v}_0, \dots, \tilde{v}_{T-1}$.

For a given k there exists only a finite number of distinct $(k(2^{k+1}+1), 2k+1)$-operators. For each of them choose a definite circuit C realizing it, made of functional elements, and consider the network in Fig. 1*a* with the chosen C, k delays, $m = k \cdot 2^{k+1}$ inputs, and $n = k+1$ outputs. The networks obtained will be called **standard *k*-networks**. It is clear that every network with $k \cdot 2^{k+1}$ inputs, $k+1$ outputs, and k delays is almost identical with some standard k-network.

Let us now consider the system Σ_k of transformation rules formed by all possible pairs of equivalent standard k-networks. The system Σ_k is finite. Let A and B

be equivalent networks with k delays. Construct, as before, the networks P and Q. P can be obtained from A, and Q from B, by the rules S_0, since A and P, as well as B and Q, are almost identical. Select in P a subnetwork P', and in Q a subnetwork Q'. There exist standard k-networks P'' and Q'', almost identical with P' and Q', respectively. By Lemma 1 these networks are equivalent and therefore form one of the rules of the system Σ_k. P' can be obtained from Q' with the aid of this rule and the rules S_0. The same is true for P and Q: indeed, P is obtained from Q by replacing Q' by P'. Thus, A can be transformed into B by the rules of the finite system $S_0 \cup \Sigma_k$ (depending only on k). The theorem is proved.

4. Let A be a network without inputs, with N delay elements, whose outputs we denote by $z_1, \dots, z_N$. We shall call the state of the network A at time t the vector $\tilde{z}_t$ of the states of the delay outputs at that moment ($\tilde{z}_0 = \tilde{0}$). The sequence $\tilde{z}_0, \tilde{z}_1, \dots$ is (ultimately) periodic; denote the length of its minimal period by T_A.

Lemma 2. *Let A be a network without inputs, and let B be the network obtained from A by replacing a subnetwork C by an equivalent one. Let k be the number of delays of the network C, and let p be a prime number greater than 2^k. If T_A is divisible by p, then T_B is divisible by p.*

Proof. Let $\widetilde{z'}^{A} = (z_1^A, \dots, z_k^A)$ be the vector of the outputs of the delays of the network A that belong to C; $\tilde{z}''^{A} = (z_{k+1}^A, \dots, z_N^A)$ the outputs of the remaining delays of the network A; and $\tilde{z}''^{B}$ the "same" vertices of the network B. Since B is obtained from A by replacing the subnetwork C by an equivalent one, and the vertices $\tilde{z}''^{A}$ are "not affected" by this replacement, we have

$$\tilde{z}_t''^{A} = \tilde{z}_t''^{B} \quad (t = 0, 1, \dots).$$

Let τ be the length of the minimal period of the sequence

$$\Pi = \{\tilde{z}_t''^{A}\} = \{\tilde{z}_t''^{B}\}.$$

The period T_A of the sequence $\{\tilde{z}_t^A\}$ is a (not necessarily minimal) period of the sequence Π; hence T_A is divisible by τ; in exactly the same way T_B is divisible by τ.

The state $\tilde{z}_t^A$ of the network A is obtained from $\tilde{z}_t''^{A}$ by adding k coordinates $z_1, z_2, \dots, z_k$. Therefore

$$T_A \leq \tau \cdot 2^k$$

(since in the minimal period of the sequence $\{\tilde{z}_t^A\}$ all vectors are distinct). It follows that the natural number

$$a = T_A / \tau$$

is less than 2^k, and hence a is not divisible by p, since $p > 2^k$. But, by assumption, T_A is divisible by p. Consequently, τ, and therefore also the multiple T_B of it, is divisible by p. The lemma is proved.

Theorem 2. *Whatever finite system of rules S is taken, there exist two equivalent networks which cannot be transformed into one another by the rules of the system S.*

This theorem follows easily from Lemma 2.

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Note: Figure translations are in progress. See original paper for figures.