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# MATHEMATICS

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**Abstract**

**Full Text**

**MATHEMATICS**

**A. P. OSKOLKOV**

## ON THE SOLUTION OF BOUNDARY VALUE PROBLEMS FOR LINEAR ELLIPTIC EQUATIONS IN AN INFINITE DOMAIN

*(Presented by Academician V. I. Smirnov on 31 V 1963)*

1. Consider, in a three-dimensional domain  $\Omega$  lying outside a closed Lyapunov surface  $S$  of class  $L_2(1, \alpha)$ ,  $0 < \alpha < 1$ , the linear uniformly elliptic equation

$$Lu \equiv a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + a_i(x) \frac{\partial u}{\partial x_i} + a(x)u = f(x), \quad x = (x_1, x_2, x_3); \quad (1)$$

$$\lambda_1 \sum_{i=1}^3 \xi_i^2 \leq a_{ij}(x) \xi_i \xi_j \leq \lambda_2 \sum_{i=1}^3 \xi_i^2, \quad \lambda_1 > 0. \quad (2)$$

Let us introduce several definitions. We shall call a function  $\delta(t)$ ,  $0 < a \leq t < \infty$ , a **Dini function** if for it there exists and is finite the integral

$$\int_a^\infty \frac{\delta(t)}{t} dt < \infty. \quad (3)$$

Let  $\delta(t)$  be a Dini function. Introduce the function

$$I_\delta(\xi) \equiv \int_\xi^\infty \frac{\delta(t)}{t} dt, \quad a \leq \xi < \infty. \quad (4)$$

Obviously,  $I_\delta(\xi) \rightarrow 0$  as  $\xi \rightarrow \infty$ .

Denote by  $\Delta(\varepsilon)$ ,  $\varepsilon > 0$ , the class of Dini functions  $\delta(t)$ ,  $0 < a \leq t < \infty$ , satisfying the following conditions:

- 1)  $\delta(t)$  is a positive, monotonically decreasing, continuously differentiable Dini function;
- 2)  $|\delta'(t)| \leq \varepsilon \frac{\delta(t)}{t}$ .

It is obvious that if  $\delta(t) \in \Delta(\varepsilon)$ , then  $\delta(t) \geq t^{-\varepsilon}$ . Further, if  $0 < \varepsilon_1 \leq \varepsilon_2$ , then  $\Delta(\varepsilon_1) \supseteq \Delta(\varepsilon_2)$ .

We shall assume that the origin of coordinates lies inside the surface  $S$ , and let  $|\tilde{x}| \equiv \max\{|x|, |x'|\}$ ,  $x, x' \in \bar{\Omega}$ . Suppose that the coefficients of equations (1)–(2) satisfy the following conditions:

$$\|a_{ij}\|_{C^{(0,\alpha)}(\bar{\Omega})} \equiv \max_{x \in \bar{\Omega}} \sum_{i,j=1}^3 |a_{ij}(x)| + \max_{x,x' \in \bar{\Omega}} \left\{ |\tilde{x}|^\alpha \sum_{i,j} \frac{|a_{ij}(x) - a_{ij}(x')|}{|x - x'|^\alpha} \right\} \leq K, \quad (5)$$

$$\|a_i\|_{C^{(0,\alpha)}_1(\bar{\Omega})} \equiv \max \left\{ |x| \sum_{i=1}^3 |a_i(x)| \right\} + \max \left\{ |\tilde{x}|^{1+\alpha} \sum_i \frac{|a_i(x) - a_i(x')|}{|x - x'|^\alpha} \right\} \leq K,$$

$$a(x) \leq 0, \quad \|a\|_{C^{(0,\alpha)}_{2,\delta/I_\delta}(\Delta, \bar{\Omega})} \equiv$$

$$\equiv \max_{x \in \bar{\Omega}} \left\{ \frac{I_\delta(|x|)}{\delta(|x|)} |x|^2 |a(x)| \right\} + \max_{x,x' \in \bar{\Omega}} \left\{ \frac{I_\delta(|\tilde{x}|)}{\delta(|\tilde{x}|)} |\tilde{x}|^{2+\alpha} \frac{|a(x) - a(x')|}{|x - x'|^\alpha} \right\} \leq K. \quad (6)$$

We shall say that the free term  $f(x)$  belongs to the class  $C^{(0,\alpha)}_{2,\delta}(\Delta, \bar{\Omega})$ ,  $0 < \alpha < 1$ ,  $\delta(t) \in \Delta(\varepsilon)$ , if the following norm is defined and finite for it:

$$\|f\|_{C^{(0,\alpha)}_{2,\delta}(\Delta, \bar{\Omega})} \equiv \max_{x \in \bar{\Omega}} \left\{ \frac{|x|^2}{\delta(|x|)} |f(x)| \right\} + \max_{x,x' \in \bar{\Omega}} \left\{ \frac{|\tilde{x}|^{2+\alpha}}{\delta(|\tilde{x}|)} \frac{|f(x) - f(x')|}{|x - x'|^\alpha} \right\} < \infty. \quad (7)$$

We shall further say that the function  $u(x)$  belongs to the class  $C^{(l,\alpha)}_{\delta,l_\delta}(\Delta, \bar{\Omega})$ ,  $l = 0, 1, 2$ ;  $0 < \alpha < 1$ ;  $\delta(t) \in \Delta(\varepsilon)$ , if the following norm is defined and finite for it:

$$\begin{aligned} \|u\|_{C^{(l,\alpha)}_{\delta,l_\delta}(\Delta, \bar{\Omega})} &\equiv \max_{x \in \bar{\Omega}} \frac{|u(x)|}{l_\delta(|x|)} + \max_{x,x' \in \bar{\Omega}} \left\{ \frac{|\tilde{x}|^\alpha}{\delta(|\tilde{x}|)} \frac{|u(x) - u(x')|}{|x - x'|^\alpha} \right\} \\ &+ \sum_{k=1}^l \max \left\{ \frac{|x|^k}{\delta(|x|)} \sum_{i,j=1}^3 |D_{ij}^k u(x)| \right\} + \\ &+ \sum_{k=1}^l \max \left\{ \frac{|\tilde{x}|^{k+\alpha}}{\delta(|\tilde{x}|)} \sum_{i,j} \frac{|D_{ij}^k u(x) - D_{ij}^k u(x')|}{|x - x'|^\alpha} \right\} < \infty. \end{aligned} \quad (8)$$

It is obvious that if the coefficients of the operator  $L$  satisfy conditions (5)–(6) and  $u(x) \in C^{(2,\alpha)}_{\delta,l_\delta}(\Delta, \bar{\Omega})$ , then  $Lu(x) \in C^{(0,\alpha)}_{2,\delta}(\Delta, \bar{\Omega})$ .

Let us consider, for equation (1)–(2), the Dirichlet problem with zero boundary condition at infinity:

$$\begin{aligned} Lu = f(x) \quad \text{in } \Omega, \quad f(x) &\in C_{2,\delta}^{(0,\alpha)}(\Delta, \bar{\Omega}), \\ u|_S = \varphi(x), \quad \varphi(x) &\in C^{(2,\alpha)}(S), \\ u(x) \rightarrow 0, \quad |x| &\rightarrow \infty. \end{aligned} \quad (9)$$

For solutions of problem (9) from the class  $C_{\delta,l_\delta}^{(2,\alpha)}(\Delta, \bar{\Omega})$  the following a priori estimates of Schauder type are valid.

**Theorem 1.** If the coefficients of equation (1)–(2) satisfy conditions (5), (6), then for any solution of the boundary-value problem (9) from the class  $C_{\delta,l_\delta}^{(2,\alpha)}(\Delta, \bar{\Omega})$  the following inequality is valid:

$$\|u\|_{C_{\delta,l_\delta}^{(2,\alpha)}(\Delta, \bar{\Omega})} \leq C(\lambda_1, K) \left[ \|f\|_{C_{2,\delta}^{(0,\alpha)}(\Delta, \bar{\Omega})} + \|\varphi\|_{C^{(2,\alpha)}(S)} + \max_{x \in \bar{\Omega}} \frac{|u(x)|}{l_\delta(|x|)} \right]. \quad (10)$$

**Theorem 2.** If the coefficients of equation (1)–(2) satisfy conditions (5), (6) and the condition

$$\min_{x \in \bar{\Omega}} \left[ \sum_{i=1}^3 a_{ii}(x) + a_i(x)x_i - (2 + \varepsilon) \frac{a_{ij}(x)x_i x_j}{|x|^2} \right] \equiv K_1(\varepsilon) > 0, \quad (11)$$

where  $\varepsilon > 0$  is the same as in the definition of the class  $\Delta(\varepsilon)$ , according to which the class of free terms  $C_{2,\delta}^{(0,\alpha)}(\Delta, \bar{\Omega})$  is constructed, then for any solution of problem (9) from the class  $C_{\delta,l_\delta}^{(2,\alpha)}(\Delta, \bar{\Omega})$  the inequality holds:

$$\|u\|_{C_{\delta,l_\delta}^{(2,\alpha)}(\Delta, \bar{\Omega})} \leq \frac{C(\lambda_1, K)}{K_1(\varepsilon)} \left[ \|f\|_{C_{2,\delta}^{(0,\alpha)}(\Delta, \bar{\Omega})} + \|\varphi\|_{C^{(2,\alpha)}(S)} \right]. \quad (12)$$

An example of an equation

$$\Delta u = x^{-3}, \quad |x| \geq 1; \quad u(x) = \frac{\ln|x|}{|x|} \equiv l_\delta(|x|) \ln|x|, \quad (13)$$

shows that if  $K_1(\varepsilon) = 0$ , then the solution of equation (1)–(2), (5), (6) with right-hand side from the class  $C_{2,\delta}^{(0,\alpha)}(\Delta, \bar{\Omega})$  may tend to zero as  $|x| \rightarrow \infty$  more slowly than  $I_\delta(|x|)$ .

Let us note that condition (11) essentially imposes a restriction on the spread of the eigenvalues of the quadratic form  $a_{ij}(x)\xi_i\xi_j$ .

Let  $0 < \varepsilon < 1$ . Then condition (11) is also fulfilled for the Laplace operator (for it  $K_1(\varepsilon) = 1 - \varepsilon$ ). It is easy to see that for the Laplace operator problem (9) is solvable in the class  $C_{\delta, I_\delta}^{(2, \alpha)}(\Delta, \bar{\Omega})$ , and then, using the a priori estimate (12), by the method of continuation with respect to a parameter we prove the theorem on solvability of the boundary-value problem (9) in the class  $C_{\delta, I_\delta}^{(2, \alpha)}(\Delta, \bar{\Omega})$  also for equation (1)–(2).

**Theorem 3.** *If the coefficients of equation (1)–(2) satisfy conditions (5), (6), (11), with  $0 < \varepsilon < 1$ , then the boundary-value problem (9) has a unique solution in the class  $C_{\delta, I_\delta}^{(2, \alpha)}(\Delta, \bar{\Omega})$ .*

Theorems 1–3 in the case  $\delta(t) = t^{-\varepsilon}$ ,  $0 < \varepsilon < 1$ , were proved in the author's paper (1).

Let now  $\varepsilon \geq 1$ . Obviously, if  $0 < \beta \leq \varepsilon$ , then

$$\Delta(\beta) \supseteq \Delta(\varepsilon), \quad K_1(\beta) \geq K_1(\varepsilon), \quad C_{2, \delta}^{(0, \alpha)}(\Delta(\beta), \bar{\Omega}) \supseteq C_{2, \delta}^{(0, \alpha)}(\Delta(\varepsilon), \bar{\Omega}),$$

$$C_{\delta, I_\delta}^{(2, \alpha)}(\Delta(\beta), \bar{\Omega}) \supseteq C_{\delta, I_\delta}^{(2, \alpha)}(\Delta(\varepsilon), \bar{\Omega}).$$

Therefore the following theorem follows immediately from Theorem 3.

**Theorem 4.** *If the coefficients of equation (1)–(2) satisfy conditions (5), (6), (11), with  $\varepsilon \geq 1$ , then the boundary-value problem (9) has a unique solution in the class*

$$\bigcap_{0 < \beta < 1} C_{\delta, I_\delta}^{(2, \alpha)}(\Delta(\beta), \bar{\Omega}).$$

**2.** The solvability conditions for the boundary-value problem (9) in the class  $C_{\delta, I_\delta}^{(2, \alpha)}(\Delta, \bar{\Omega})$  can be considerably broadened. Suppose that the coefficients of equation (1)–(2) satisfy conditions (5), (6), (11),  $0 < \varepsilon < 1$ ,  $f(x) \in C_{2, \delta}^{(0, \alpha)}(\Delta, \bar{\Omega})$ , and first consider the following spectral problem:

$$Lu = f(x) + \lambda q(x)u \text{ in } \Omega, \quad q(x) \in C_{2, \delta/I_\delta}^{(0, \alpha)}(\Delta, \bar{\Omega}), \quad (14)$$

$$u|_S = 0; \quad u(x) \rightarrow 0, \quad |x| \rightarrow \infty.$$

From Theorem 3 and the properties of  $q(x)$  it follows that the operator  $\mathfrak{M}_0 u \equiv L^{-1}(qu)$  is completely continuous in the space  $C_{\delta, I_\delta}^{(0, \alpha)}(\Delta, \bar{\Omega})$ . Therefore problem (14) reduces to the operator equation  $u = F + \lambda_0 \mathfrak{M}_0 u$ ,  $F \equiv L^{-1}f$ , in the space  $C_{\delta, I_\delta}^{(0, \alpha)}(\Delta, \bar{\Omega})$  with the completely continuous operator  $\mathfrak{M}_0$ . The spectrum of such an equation, and hence the spectrum of problem (14), is discrete and consists of at most a countable set of points in the complex  $\lambda$ -plane, with a possible

accumulation point at infinity. If  $\lambda$  is not equal to any of these exceptional values, then the boundary-value problem (14) is solvable in the class  $C_{\delta, I_\delta}^{(2, \alpha)}(\Delta, \bar{\Omega})$  for any  $f(x) \in C_{2, \delta}^{(0, \alpha)}(\Delta, \bar{\Omega})$ .

We shall obtain a significant extension of Theorem 3 by considering the following spectral problem:

$$Lu = f(x) + \mu A_i(x) \frac{\partial u}{\partial x_i} \text{ in } \Omega, \quad A_i(x) \in C_1^{(0, \alpha)}(\bar{\Omega}), \quad (15)$$

$$u|_S = 0; \quad u(x) \rightarrow 0, \quad |x| \rightarrow \infty.$$

From Theorem 3 and the properties of the coefficients  $A_i(x)$  it follows that the operator

$$\mathfrak{M}_1 u = L^{-1} \left( A_i \frac{\partial u}{\partial x_i} \right)$$

is completely continuous in the space  $C_{\delta, l/\delta}^{(1, \alpha)}(\Delta, \bar{\Omega})$ . Therefore problem (15) reduces to the operator equation  $u = F + \mu \mathfrak{M}_1 u$  in the space  $C_{\delta, l/\delta}^{(1, \alpha)}(\Delta, \bar{\Omega})$  with a completely continuous operator  $\mathfrak{M}_1$ . Hence it follows that the spectrum of problem (15) is discrete and consists of no more than a countable set of points in the complex plane ( $\mu$ ), with a possible accumulation point at infinity.

Obviously, if

$$\min_{x \in \bar{\Omega}} \left[ \sum_{i=1}^3 a_{ii}(x) + (a_i - \mu A_i) x_i - (2 + \varepsilon) \frac{a_{ij} x_i x_j}{|x|^2} \right] \equiv K_1(\varepsilon, \mu) > 0, \quad (16)$$

then  $\mu$  is not a point of the spectrum of problem (15). The example of the equation

$$\Delta u + \frac{l-2}{3} \frac{1}{x_i} \frac{\partial u}{\partial x_i} = |x|^{-(l+1)}, \quad 1 < l < 2, \quad |x| \geq 1;$$

$$u(x) = O(I_\delta(|x|) \ln |x|), \quad |x| \rightarrow \infty, \quad (17)$$

shows that if  $K_1(\varepsilon, \mu) = 0$ , then  $\mu$  can already be a point of the spectrum of problem (15).

3. Analogous results are valid for the boundary-value problem with normal derivative

$$Lu = f(x) \text{ in } \Omega, \quad f(x) \in C_{2,\delta}^{(0,\alpha)}(\Delta, \bar{\Omega}),$$

$$\left( \frac{\partial u}{\partial n} + B(x)u \right) \Big|_S = \psi(x), \quad B(x) \leq 0, \quad B(x), \psi(x) \in C^{(1,\alpha)}(S), \quad (18)$$

$$u(x) \rightarrow 0, \quad |x| \rightarrow \infty.$$

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## REFERENCES

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*Note: Figure translations are in progress. See original paper for figures.*

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