



Soviet-era science, translated into English

MATHEMATICS

1963

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196301.53106>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

MATHEMATICS

S. P. NOVIKOV

HOMOTOPIC PROPERTIES OF THE GROUP OF DIFFEOMORPHISMS OF THE SPHERE

(Presented by Academician L. S. Pontryagin on 30 VI 1962)

By $\text{diff } M^n$ and $\text{diff}^0 M^n$ we shall denote, respectively, the group of diffeomorphisms of a smooth manifold M^n preserving orientation, and its linearly connected component of the identity in the C^∞ -topology. Let W be a smooth manifold of class C^∞ .

Definition 1. A mapping $f : W \rightarrow \text{diff } M^n$ will be called smooth of class C^r ($r \geq 0$) if the mapping $F(f) : W \times M^n \rightarrow W \times M^n$ such that

$$F(f)(x, y) = (x, f_x(y)),$$

where $x \in W$, $y \in M^n$, is a diffeomorphism of class C^r .

The following is easily proved.

Lemma 1. *Every mapping $f : W \rightarrow \text{diff } M^n$ is approximated arbitrarily closely by a smooth mapping of class C^∞ .*

Our ultimate aim is to study the groups

$$\text{diff } S^{n-1}, \quad \text{diff}^0 S^{n-1}, \quad \text{diff } D^n, \quad \text{diff}^0 D^n, \quad \overline{K}^n = K^n \cap \text{diff}^0 D^n,$$

where K^n is the group of diffeomorphisms of the ball D^n , fixed on the boundary $S^{n-1} = \partial D^n$. A fibered space in the sense of Serre is defined:

$$\text{diff}^0 D^n \xrightarrow[p]{\overline{K}^n} \text{diff}^0 S^{n-1},$$

where p is the natural projection. The orthogonal group SO_n is naturally imbedded in the group $\text{diff}^0 S^{n-1}$. Milnor (6-8) established, for a number of dimensions n , the nontriviality of the groups $\pi_0(\text{diff } S^{n-1})$, and even of the group $\pi_0(\text{diff } S^{n-1})/p_*\pi_0(\text{diff } D^n)$, which has been more or less completely computed for $n \geq 6$. We shall study the groups $\pi_i(\text{diff}^0 S^{n-1})/p_*\pi_i(\text{diff}^0 D^n)$ for a number of values of n and for $i > 0$. For this purpose we shall first study the relation between the Whitehead homomorphism $J : \pi_i(SO_N) \rightarrow \pi_{N+i}(S^N)$ and the composition multiplication in the ring

$$G = \sum_{i \geq 0} G_i$$

of stable homotopy groups of spheres $G_i = \pi_{N+i}(S^N)$, $N > i+1$. By $J_i \subset G_i$ we shall denote the image $J\pi_i(SO_N)$, and by $\tilde{\theta}_i \subset G_i$ we shall denote the subgroup of the group G_i each element of which is represented by a framed manifold combinatorially equivalent to the sphere S^i . Obviously $J_i \subset \tilde{\theta}_i$. It is known that $\tilde{\theta}_i = G_i$ for $i \not\equiv 2 \pmod{4}$ or for $i = 10$, and that $G_i/\tilde{\theta}_i$ contains exactly 2 elements for $i = 2, 6, 14$ and no more than 2 elements in the remaining cases (these results are due to Kervaire, Milnor, and Smale ^(5,6,10,11)). Using the technique of Morse modifications and surgeries of framed manifolds ^(3,5), we prove comparatively simply the following theorem.

Theorem 1. *Let $\alpha \in G_i$, $\beta \in G_j$, $i > 0$, $j > 0$. Then $\alpha \circ \beta \in \tilde{\theta}_{i+j}$ for all pairs (i, j) , except for the cases $i = j = 1, 3, 7$.*

As is known, the Morse surgery depends on the embedding of the sphere $S^i \subset M^n$ with trivial normal bundle and on an element $h \in \pi_i(SO_{n-i})$. As a result of the Morse surgery we arrive at the manifold $M^n(S^i, h)$.

It is not hard to prove

Lemma 2. *Let $M^n = S^i \times S^j$. Then the manifold $M^n(S^i, h)$ is diffeomorphic to S^n .*

The proof is based on the fact that every diffeomorphism $\tilde{h}: S^i \times D^j \rightarrow S^i \times D^j$ such that $\tilde{h}(x, y) = (x, \tilde{h}_x(y))$ extends to a diffeomorphism $\tilde{h}: S^i \times S^j \rightarrow S^i \times S^j$ such that $\tilde{h}/S^i \times D^j = (\tilde{h}_x \in SO_j)$.

From Lemma 2 it follows that

Lemma 3. *Let $\alpha \in J_i$, $\beta \in J_j$. Then $\alpha \circ \beta \in J_{i+j}$, except in the cases $i = j = 1, 3, 7$.*

We shall now denote by $B(M^n) \subset G_n$, for a π -manifold M^n , the set of elements $\alpha \in B(M^n) \subset G_n$ having as a representative the manifold M^n with some framing. With the aid of Morse surgeries, Lemma 2 gives

Lemma 4. *The formula holds*

$$B(S^i \times S^j) = J_{i+j},$$

except in the cases $i = j = 1, 3, 7$.

From the results of Haefliger ^(1,2), Smale ⁽¹¹⁾, and Kervaire ⁽⁴⁾ it is not difficult to derive

Lemma 5. *Let \tilde{S}^j be a smooth π -manifold homeomorphic to a sphere, and let $i > j/2 + 1$. Then the manifold $S^i \times \tilde{S}^j$ is diffeomorphic to the direct product $S^i \times S^j$.*

The proof of Lemma 5 is obtained from Haefliger' s theorems on approximation of topological embeddings by smooth ones, Smale' s theorems on J -equivalence,

and Kervaire's theorem on normal bundles of homotopy spheres in Euclidean space.

From Lemmas 2-5 it follows that

Theorem 2. *Let $\alpha \in J_i$, $\beta \in \tilde{\theta}_j$, with $i > j/2 + 1$. Then $\alpha \circ \beta \in J_{i+j}$, except in the cases $i = j = 1, 3, 7$.*

In the case when the hypotheses of Theorem 2 are not fulfilled, i.e. if $i \leq j/2 + 1$, a situation is possible in which $J_i \circ \tilde{\theta}_j \not\subset J_{i+j}$ for suitably chosen dimensions i, j . Let $\beta \in \tilde{\theta}_j$, and let the element β be represented by a framed homotopy sphere \tilde{S}_β^j with some framing; moreover the sphere \tilde{S}_β^j is determined uniquely modulo $\theta^j(\partial\pi)$. As is known, the sphere \tilde{S}_β^j decomposes into the union of two disks

$$\tilde{S}_\beta^j = D^j \cup_{q_\beta} D^j,$$

where $q_\beta \in \text{diff } S^{j-1}$, i.e. $q_\beta \in \pi_0(\text{diff } S^{j-1})$, and determines an element

$$\tilde{q}_\beta \in \pi_0(\text{diff } S^{j-1})/p_*\pi_0(\text{diff } D^j).$$

The following lemma is very important for our purposes.

Lemma 6. *Let $\alpha \in J_i$, $\alpha \circ \beta \notin J_{i+j}$. Then there exists a smooth map $h : S^i \rightarrow SO_j$ such that the diffeomorphism $F_\beta(h)(x, y) = (x, [q_\beta h_x q_\beta^{-1}](y))$ does not extend to a diffeomorphism $S^i \times D^j \rightarrow S^i \times D^j$, where*

$$F_\beta(h) : S^i \times S^{j-1} \rightarrow S^i \times S^{j-1}.$$

Proof. Consider the direct product $S^i \times \tilde{S}_\beta^j$ and specify on it two different framings: the trivial one and the framing corresponding to the element $\alpha \circ \beta \notin J_{i+j}$. Perform two Morse surgeries and pass to the manifolds $M^n(S^i, h_1)$ and $M^n(S^i, h_2)$, where $h_i : S^i \rightarrow SO_j$ and $n = i + j$. Choose the map h_1 so that it would be possible to drag ...

on $M^n(S^i, h_1)$ a trivial framing, and h_2 so that on $M^n(S^i, h_2)$ one could drag the framing defining the element $\alpha \circ \beta$. Both manifolds $M^n(S^i, h_1)$ and $M^n(S^i, h_2)$ are homotopy spheres, but $M^n(S^i, h_1) \in \theta^n(\partial\pi)$, while $M^n(S^i, h_2) \notin \theta^n(\partial\pi)$. Therefore the map $h = h_1 h_2^{-1}$ is such that the diffeomorphism

$$F(h) : S^i \times D^j \rightarrow S^i \times D^j,$$

where $F(h)(x, y) = (x, h_x(y))$, does not extend to a diffeomorphism

$$S^i \times \tilde{S}_\beta^j \rightarrow S^i \times \tilde{S}_\beta^j.$$

This is equivalent to saying that the diffeomorphism

$$F_\beta(h) : S^i \times S^{j-1} \rightarrow S^i \times S^{j-1},$$

where $F_\beta(h)(x, y) = (x, [q_\beta h_x q_\beta^{-1}](y))$, does not extend to $S^i \times D^j$. Thus the lemma is proved.

From Lemmas 5 and 1 we obtain the following

Corollary 1. Let $\alpha \in J_i, \beta \in \hat{\theta}_j, \alpha \circ \beta \notin J_{i+j}$. Then there exists a diffeomorphism $q_\beta : S^{j-1} \rightarrow S^{j-1}, q_\beta \notin \text{diff}^0 S^{j-1}$, and there exists an element $h \in \pi_i(SO_j)$ such that the element

$$q_\beta h q_\beta^{-1} \in \pi_i(\text{diff}^0 S^{j-1})$$

does not belong to

$$\rho_* \pi_i(\text{diff}^0 D^j)$$

(one may assume that $q_\beta \in \pi^0(\text{diff} S^{j-1})$).

For the application of the last results it is necessary to know the structure of the groups G_i , the products $G_i \circ G_j$, the image $\text{Im } J$, and the subgroups $\hat{\theta}_i$. We give a table of these groups for $i \leq 14$ and a multiplication table for $G_i \circ G_j$.

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14
G_i	Z_2	Z_2	Z_{24}	0	0	Z_2	Z_{240}	Z_2+ Z_2	Z_2+ Z_2+ Z_2	Z_6	Z_{504}	0	Z_3	Z_2
$\tilde{\theta}_i$	Z_2	0	Z_{24}	0	0	0	Z_{240}	Z_2+ Z_2	Z_2+ Z_2+ Z_2	Z_6	Z_{504}	0	Z_3	0
J_i	Z_2	0	Z_{24}	0	0	0	Z_{240}	Z_2	Z_2	0	Z_{504}	0	0	0

One can choose generators

$$x_i^{(p)}, y_i^{(p)}, z_i^{(p)} \in G_i$$

(p is a prime number)

$$x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, x_3^{(3)}, x_6^{(2)}, x_7^{(2)}, x_7^{(3)}, x_7^{(5)}, x_8^{(2)}, y_8^{(2)}, x_9^{(2)}, y_9^{(2)}, z_9^{(2)}, x_{10}^{(2)}, x_{10}^{(3)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(7)}, x_{13}^{(3)}, x_{14}^{(2)},$$

such that:

1. $2x_1^{(2)} = 0, 2x_2^{(2)} = 0, 8x_3^{(2)} = 0, 3x_3^{(3)} = 0, 2x_6^{(2)} = 0, 16x_7^{(2)} = 0, 3x_7^{(3)} = 0,$
 $5x_7^{(5)} = 0, 2x_8^{(2)} = 0, 2y_8^{(2)} = 0, 2x_9^{(2)} = 0, 2y_9^{(2)} = 0, 2z_9^{(2)} = 0, 2x_{10}^{(2)} = 0,$
 $3x_{10}^{(3)} = 0, 8x_{11}^{(2)} = 0, 9x_{11}^{(3)} = 0, 7x_{11}^{(7)} = 0, 3x_{13}^{(3)} = 0, 2x_{14}^{(2)} = 0.$

2. $(x_1^{(2)})^2 = x_2^{(2)}$, $(x_1^{(2)})^3 = 4x_3^{(2)}$, $(x_3^{(2)})^2 = x_6^{(2)}$, $x_1^{(2)}x_7^{(2)} = x_8^{(2)}$, $(x_1^{(2)})^2x_7^{(2)} = (x_3^{(2)})^3 = x_9^{(2)}$,
 $x_1^{(2)}y_8^{(2)} = y_9^{(2)}$, $(x_1^{(2)})^2y_8^{(2)} = 0$, $(x_1^{(2)})^3x_7^{(2)} = 0$, $x_1^{(2)}z_9^{(2)} = x_{10}^{(2)}$, $(x_1^{(2)})^2z_9^{(2)} = 4x_{11}^{(2)}$, $(x_7^{(2)})^2 = x_{14}^{(2)}$,
 $x_{11}^{(2)}x_3^{(2)} = 0$, $x_3^{(3)}x_{10}^{(3)} = x_{13}^{(3)}$.
3. $x_1^{(2)}$, $x_3^{(2)}$, $x_3^{(3)}$, $x_7^{(2)}$, $x_7^{(3)}$, $x_7^{(5)}$, $x_8^{(2)}$, $x_9^{(2)}$, $x_{11}^{(2)}$, $x_{11}^{(3)}$, $x_{11}^{(7)} \in \text{Im } J$; the remaining generating elements do not belong to $\text{Im } J$.
4. All generators, except

$$(x_1^{(2)})^2 = x_2^{(2)}, \quad (x_3^{(2)})^2 = x_6^{(2)}, \quad (x_7^{(2)})^2 = x_{14}^{(2)},$$

belong to the subgroups $\tilde{\theta}_i$.

Further, concerning the p -components of the groups $G_i^{(p)}$, it is known that:

- 1) $G_{2p-3}^{(p)} = Z_p = J_{2p-3}^{(p)}$ (generator $x_{2p-3}^{(p)}$);
 2) $G_{2p(p-1)-2}^{(p)} = \hat{\theta}_{2p(p-1)-2}^{(p)} = Z_p$ (generator $x_{2p(p-1)-2}^{(p)}$), and for $p > 2$ the group

$$I_{2p(p-1)-2}^{(p)} = 0;$$

- 3) the elements

$$x_{2p-3}^{(p)} \circ (x_{2p(p-1)-2}^{(p)})^k \notin \text{Im } J$$

for $k \leq p - 2$, and

$$x_{2p-3}^{(p)} \circ (x_{2p(p-1)-2}^{(p)})^{p-1} \neq 0, \quad (x_{2p(p-1)-2}^{(p)})^p \neq 0.$$

(For results concerning multiplication in the homotopy groups of spheres, see (9).)*

* We note that earlier it was believed that

$$G_{14} = \pi_{N+14}(S^N) = Z_2 + Z_2.$$

However, this result is incorrect. The author showed that $G_{14} = Z_2$, using the fact that

$$J_8 \circ J_{11} \subset J_{14} = 0$$

and that

$$G_{14} = J_7 \circ J_7 \cup J_3 \circ J_{11}.$$

Now it remains for us, using the preceding results and the information on the groups G_n, θ_n, J_n and the multiplication $G_i \circ \tilde{G}_j$, to indicate cases of nontriviality of the groups $A_{i,j} = \pi_i(\text{diff}^0 S^{j-1})/p_*\pi_i(\text{diff}^0 D^j)$.

Theorem 3. *The groups*

$$A_{i,j} = \pi_i(\text{diff}^0 S^{j-1})/p_*\pi_i(\text{diff}^0 D^j)$$

have the following form:

- 1) $A_{1,8} \supset Z_2$;
- 2) $A_{1,9} \supset Z_2$;
- 3) $A_{2p-3, 2kp(p-1)-2k} \otimes Z_p \supset Z_p + \dots + Z_p$ ($p-1$ summands), $p \geq 3$ for $k \leq p-2$;
 $A_{3,10} \otimes Z_3 \supset Z_3 + Z_3$ for $p = 3$.

The proof of Theorem 3 follows at once from the lemmas and from the structure of the ring $G = \sum G_i$. Since $\pi_1(SO_n) = Z_2$ ($n > 2$), it follows that $h \in Z_2$ (see Lemma 6), and the element β has order 2, $\beta \in G_8(G_9)$. Therefore $q_\beta h q_\beta^{-1} \in \pi_1(\text{diff}^0 S^7)$ also has order 2 and $q_\beta^2 \in \text{diff}^0 S^7$. Hence the group $\pi_1(\text{diff}^0 S^7) \supset Z_2 + Z_2$ with generators h and $q_\beta h q_\beta^{-1} \notin \text{Im } p_*$. Thus $A_{1,8} \supset Z_2$. Similarly for $A_{1,9}$. Assertions 1) and 2) are proved. Let us prove assertion 3). Note that $\pi_{2p-3}(SO_j) = Z$ for $j > 2p-2$. Let h be a generator of the group $\pi_{2p-3}(SO_j)$, and $\beta = \chi(p)_{2p(p-1)-2}^k$, $k \leq p-2$. Then $h, q_\beta h q_\beta^{-1}, \dots, q_\beta^{p-1} h q_\beta^{1-p}$ are distinct elements of the group $\pi_{2p-3}(\text{diff}^0 S^{j-1})$, $j = 2kp(p-1) - 2k$, and all these elements are of infinite order. However, relations of the form

$$\lambda_1 p h = \lambda_2 p (q_\beta h q_\beta^{-1}) = \dots = \lambda_p \circ p (q_\beta^{p-1} h q_\beta^{1-p})$$

are possible, whence the desired result follows. The theorem is proved.

As usual, by B_G we shall denote the classifying space of the group G .

Corollary 2. *The classifying space $B_{\text{diff } S^{n-1}}$ is not homotopically simple for $n = 8, 9, 2kp(p-1) - 2k$, $k \leq p-2$; namely: the group π_1 acts nontrivially on the groups respectively*

$$\pi_2(B_{\text{diff } S^7}), \quad \pi_2(B_{\text{diff } S^8}), \quad \pi_{2p-2}(B_{\text{diff } S^{2kp(p-1)-2k-1}}), \quad k \leq p-2.$$

From the fibration, in the sense of Serre,

$$\text{diff}^0 D^n \xrightarrow{\overline{K}^n} \text{diff}^0 S^{n-1},$$

where

$$\overline{K}^n = K^n \cap \text{diff}^0 D^n,$$

we obtain

Corollary 3. a) *There exists a diffeomorphism $F : D^n \rightarrow D^n$ such that $F \in \text{diff}^0 D^n$, $F/\partial D^n = 1$, and F is not isotopic to the identity in the group K^n for $n = 8, 9$; b) *the groups**

$$\pi_{2p-4}(K^{2k(p-1)p-2k}) \neq 0$$

($p > 2$) for $k \leq p - 2$ (p a prime number).

Corollary 4. *There exist sphere bundles over spheres with structural group $\text{diff}^0 S^{n-1}$, not equivalent to orthogonal bundles in the group $\text{diff}^0 S^{n-1}$, but equivalent to orthogonal bundles in the group $\text{diff} S^{n-1}$.*

Steklov Mathematical Institute
Academy of Sciences of the USSR

Received
23 VI 1962

REFERENCES

1. A. Haefliger, Bull. Am. Math. Soc., 67, 1 (1961).
2. A. Haefliger, Comm. Math. Helv., 36, 1 (1961).
3. M. Kervaire, Comm. Math. Helv., 34, 4 (1960).
4. M. Kervaire, Bull. Soc. Math. France, 87, 4 (1959).
5. M. Kervaire, J. Milnor, Preprint, 1961.
6. J. Milnor, Bull. Soc. Math. France, 87, 4 (1959).
7. J. Milnor, Ann. Math., 64, 2 (1956).
8. J. Milnor, Am. J. Math., 81, 4 (1959).
9. S. P. Novikov, DAN, 128, No. 5 (1959).
10. S. Smale, Bull. Am. Math. Soc., 66, 5 (1960).
11. S. Smale, Ann. Math., Preprint, 1961.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.