MATHEMATICS

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Abstract

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MATHEMATICS

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ON THE REPRESENTATION OF THE TRANSITION FUNCTION OF A MACHINE BY ASSOCIATIVE OPERATIONS

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1. Ginzburg ([1]) calls a **machine** a triple $A = A(\mathfrak{A}, \mathfrak{X}, \delta)$ consisting of a set of states $\mathfrak{A}$, a semigroup of input signals $\mathfrak{X}$, and a transition function δ, defined on $\mathfrak{A} \times \mathfrak{X}$ with values in $\mathfrak{A}$ and satisfying the condition

$$\delta(a, x_1 x_2) = \delta[\delta(a, x_1), x_2].$$

We shall call machines in Ginzburg' s sense **right machines**. Alongside them we introduce into consideration **left machines**, by which we shall mean the same kind of triple, with the only difference that the transition function Δ is to be regarded as defined on $\mathfrak{X} \times \mathfrak{A}$ and subject to the condition (writing this function from right to left)

$$(x_1 x_2, a)\Delta = [x_1, (x_2, a)\Delta]\Delta.$$

A pair (φ, ψ) is called an **isomorphism** of the right machine $P = P(\mathfrak{A}, \mathfrak{X}, \delta)$ onto the right machine $P' = P'(\mathfrak{A}', \mathfrak{X}', \delta')$, if φ is a one-to-one mapping of $\mathfrak{A}$ onto $\mathfrak{A}'$, ψ is an isomorphism of the semigroup $\mathfrak{X}$ onto $\mathfrak{X}'$, and, moreover,

$$\varphi[\delta(a, x)] = \delta'[\varphi(a), \psi(x)].$$

The isomorphism of left machines can be defined analogously.

If, in a right (left) machine, the set of states forms a semigroup with respect to some fixed multiplication operation in it, then we shall speak of a p-**machine**. Speaking of an isomorphism (φ, ψ) of p-machines, we shall require of φ that it be an isomorphism of the semigroups of states.

2. We shall call a triple $M = (\bar{A}, \bar{\mathfrak{A}}, \bar{\mathfrak{X}})$ a **right** (**left**) m-**semigroup** if $\bar{A}$ is a semigroup, $\bar{\mathfrak{A}}$ is its subset, $\bar{\mathfrak{X}}$ is a subsemigroup, and moreover $\bar{\mathfrak{A}}\bar{\mathfrak{X}} \subseteq \bar{\mathfrak{A}}$ (respectively $\bar{\mathfrak{X}}\bar{\mathfrak{A}} \subseteq \bar{\mathfrak{A}}$) and $\{\bar{\mathfrak{A}}, \bar{\mathfrak{X}}\} = \bar{A}$, where $\{\bar{\mathfrak{A}}, \bar{\mathfrak{X}}\}$ is the subsemigroup in $\bar{A}$ generated by the sets $\bar{\mathfrak{A}}$ and $\bar{\mathfrak{X}}$. If, in particular, $\bar{\mathfrak{A}}$ is a right (left) ideal, then the m-semigroup will be called an **ideal right** (**left**) m-**semigroup**. If $\bar{\mathfrak{A}}$ is a two-sided ideal, then the m-semigroup will be called **two-sided**.

If a right m-semigroup $M = (\bar{A}, \bar{\mathfrak{A}}, \bar{\mathfrak{X}})$ is given, then, putting

$$\delta_M(\bar{a}, \bar{x}) = \bar{a}\bar{x}, \quad \bar{x} \in \bar{\mathfrak{X}},\ \bar{a} \in \bar{\mathfrak{A}},$$

we obtain a right machine $P_M = P_M(\bar{\mathfrak{A}}, \bar{\mathfrak{X}}, \delta_M)$, of which we shall say that it is **generated by the m-semigroup** M; analogously, if M is a left m-semigroup, then it **generates a left machine** $L = L(\bar{\mathfrak{A}}, \bar{\mathfrak{X}}, \Delta)$, where

$$(\bar{x}, \bar{a})\Delta = \bar{x}\bar{a}.$$

We shall agree to say that a given right (left) m-semigroup M **represents** the right (left) machine A, if A is isomorphically mapped onto the machine generated by the m-semigroup M. In other words, a right m-semigroup $M = (\bar{A}, \bar{\mathfrak{A}}, \bar{\mathfrak{X}})$ represents the right machine $P = P(\mathfrak{A}, \mathfrak{X}, \delta)$ if there exists a pair (φ, ψ) consisting of a one-to-one mapping φ of $\mathfrak{A}$ onto $\bar{\mathfrak{A}}$ and an isomorphism ψ of $\mathfrak{X}$ onto $\bar{\mathfrak{X}}$, for which the equality

$$\delta(a, x) = \varphi^{-1}[\varphi(\bar{a})\psi(\bar{x})]$$

holds. Analogously for left m-semigroups and machines.

An **isomorphism of two m-semigroups** is an isomorphism of the machines generated by them. In order for two m-semigroups to represent isomorphic machines, it is necessary and sufficient that they themselves be isomorphic. Ideal m-semigroups represent p-machines, and conversely: if a p-machine is representable by an m-semigroup, then it is representable by an ideal one.

Proposition 1. Any right machine $A = A(\mathfrak{A}, \mathfrak{X}, \delta)$ can be isomorphically embedded in such a right p-machine $A' = A'(\mathfrak{A}', \mathfrak{X}, \delta')$, representable by a right (of course, ideal) m-semigroup, that $\mathfrak{A}'$ differs from $\mathfrak{A}$ by only one element.

Proposition 2. If a right p-machine can be represented by a right (of course, ideal) m-semigroup, then it can be isomorphically embedded in such a right p-machine that is representable by a two-sided m-semigroup.

These propositions show that in studying representations of machines by m-semigroups, attention must be paid to p-machines and to their representations by ideal (in particular, two-sided) m-semigroups.

3. The condition for representability of a right p-machine by a right m-semigroup is given by

Theorem 1. In order that a right p-machine $P = P(\mathfrak{A}, \mathfrak{X}, \delta)$ be representable by an (ideal) right m-semigroup, it is necessary and sufficient that the function δ be left-homogeneous with respect to the states, i.e., that the equality

$$\delta(a_1 a_2, x) = a_1 \delta(a_2, x).$$

hold.

4. To obtain the conditions for representability of a p-machine by a two-sided m-semigroup, several lemmas are proved. The principal one is the following (we recall the necessary definitions). If for an m-semigroup $M = (\bar{A}, \bar{\mathfrak{A}}, \bar{\mathfrak{X}})$ one of the conditions holds: $\bar{A} = \bar{\mathfrak{A}} \cup \bar{\mathfrak{X}}$, $\bar{\mathfrak{A}} \cap \bar{\mathfrak{X}} = \emptyset$, $\bar{a}\bar{x} = \bar{x}\bar{a}$ ($\bar{a} \in \bar{\mathfrak{A}}, \bar{x} \in \bar{\mathfrak{X}}$), then M is called respectively **minimal**, **separated**, or **commutative**. An isomorphism (φ, ψ) of a minimal m-semigroup $(A, \mathfrak{A}, \mathfrak{X})$ onto a minimal m-semigroup $(A', \mathfrak{A}', \mathfrak{X}')$ is called a **strong isomorphism** if, for any $\bar{x} \in \bar{\mathfrak{X}}$, $\bar{a} \in \bar{\mathfrak{A}}$, the following condition holds:

$$\bar{x}\bar{a} = \begin{cases} \varphi\left[\psi^{-1}(\bar{x})\varphi^{-1}(\bar{a})\right], & \text{if } \psi^{-1}(\bar{x})\varphi^{-1}(\bar{a}) \in \mathfrak{A}, \\ \psi\left[\psi^{-1}(\bar{x})\varphi^{-1}(\bar{a})\right], & \text{if } \psi^{-1}(\bar{x})\varphi^{-1}(\bar{a}) \notin \mathfrak{A} \end{cases}$$

(but then necessarily $\psi^{-1}(\bar{x})\varphi^{-1}(\bar{a}) \in \mathfrak{X}$, so that ψ is applicable).

Lemma. In order that a minimal right ideal m-semigroup $M = (A, \mathfrak{A}, \mathfrak{X})$ be strongly isomorphically mapped onto some separated minimal right ideal m-semigroup, it is necessary and sufficient that, for the set $I = \mathfrak{X}\mathfrak{A} \setminus (\mathfrak{X}\mathfrak{A} \cap \mathfrak{A})$, the inclusions $\mathfrak{X}I \subseteq I$, $I\mathfrak{X} \subseteq I$ hold.

5. The condition for representability of a right machine by a two-sided m-semigroup is given by

Theorem 2. In order that a right p-machine $P(\mathfrak{A}, \mathfrak{X}, \delta)$ be representable by a two-sided m-semigroup, it is necessary and sufficient that there exist such a left machine $L(\mathfrak{A}, \mathfrak{X}, \Delta)$ that

$$\delta(a_1a_2, x) = a_1\delta(a_2, x); \qquad \delta(a_1, x)a_2 = a_1(x, a_2)\Delta;$$
$$(x, a_1a_2)\Delta = (x, a_1)\Delta a_2; \qquad \delta\big[(x_1, a)\Delta, x_2\big] = [x_1, \delta(a_2, x)]\Delta.$$

In this case the representing two-sided m-semigroup can be constructed in the set $\bar{A} = \bar{\mathfrak{A}} \cap \bar{\mathfrak{X}}$, where $\bar{\mathfrak{A}}$ and $\bar{\mathfrak{X}}$ are arbitrary isomorphic

the images of the semigroups $\mathfrak{A}$ and $\mathfrak{X}$: $\overline{\mathfrak{A}} = \varphi(\mathfrak{A})$, $\overline{\mathfrak{X}} = \psi(\mathfrak{X})$, $\overline{\mathfrak{A}} \cap \overline{\mathfrak{X}} = \emptyset$, with the operation defined by:

$$\bar{a}_1\bar{a}_2 = \varphi\,[\varphi^{-1}(\bar{a}_1)\varphi^{-1}(\bar{a}_2)]; \qquad \bar{x}_1\bar{x}_2 = \psi\,[\psi^{-1}(\bar{x}_1)\psi^{-1}(\bar{x}_2)];$$
$$\bar{a}\bar{x} = \varphi\,[\delta(\varphi^{-2}(\bar{a}), \psi^{-1}(\bar{x}))]; \qquad \bar{x}\bar{a} = \varphi\,[(\psi^{-1}(\bar{x}), \varphi^{-1}(\bar{a}))\,\Delta].$$

6. From Theorem 2 there follows without difficulty

Theorem 3. In order that a right p-machine $P(\mathfrak{A}, \mathfrak{X}, \delta)$ be representable by a commutative m-semigroup, it is necessary and sufficient that its function δ satisfy the conditions:

$$\delta(a_1, x)a_2 = \delta(a_1a_2, x) = a_1\delta(a_2, x);$$

$$\delta(a, x_1 x_2) = \delta(a, x_2 x_1).$$

In this case the representing m-semigroup may be constructed in the same way as in Theorem 2, but with Δ replaced by δ.

7. It is not hard to verify that free automata ([2]), considered as machines, are representable by m-semigroups.

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REFERENCES

[1] S. Ginsburg, Trans. Am. Math. Soc., **96**, 400 (1960).
[2] V. M. Glushkov, UMN, **16**, no. 5 (101), 3 (1961).

Note: Figure translations are in progress. See original paper for figures.