PHYSICS

1963

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Abstract

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KINETIC EQUATION OF A NONIDEAL FERMI SYSTEM

(Presented by Academician N. N. Bogolyubov, 11 September 1962)

1. With the aid of Bogolyubov' s method of the generating functional (1), generalized to the case of quantum statistics (3), for the s-particle density operators $F_s = \mathrm{Sp}_{(s+1,\dots,N)}\,\rho(1,\dots,N)$ (ρ is the density matrix) a chain of coupled equations is obtained:

$$\frac{\partial F_s}{\partial t} = \left[\sum_{i=1}^{N} T_i + \sum_{1\le j<r\le N} \Phi(j,r);\ F_s\right] + \frac{1}{v}\,\mathrm{Sp}_{(s+1)}\left[\sum_{1\le k\le s} \Phi(k,s+1);\ F_{s+1}\right]. \tag{1}$$

The matrix of the potential Φ of the pair interaction of particles is not assumed to be diagonal. In order to satisfy the law of conservation of momentum, we require that the matrix elements $\Phi(r_1 r_2, r_1' r_2')$ be invariant with respect to spatial translations. In this case

$$\Phi(r_1,r_2;r_1',r_2') =$$

$$= \int \widetilde{\Phi}(p_1,p_2;p_1'p_2')\exp\{i(p_1r_1+p_2r_2-p_1'r_1'-p_2'r_2')\}\,dp_1dp_2dp_1'dp_2',$$

$$\widetilde{\Phi}(p_1,p_2;p_1',p_2') = \Phi(p_1,p_2;p_1',p_2')\,\delta(p_1+p_2-p_1'-p_2'). \tag{2}$$

By virtue of the Hermiticity of the Hamiltonian and the symmetry properties of the system,

$$\Phi(p_1,p_2;p_1',p_2') = \Phi^*(p_1',p_2';p_1,p_2), \qquad \Phi(p_1,p_2;p_1',p_2') = \Phi(p_2,p_1;p_2',p_1'). \tag{3}$$

We shall consider the system of equations (1) under the assumption that the potential energy of interaction of a pair of particles is small in comparison with their mean kinetic energy. In this case a small parameter of perturbation theory $^{(2)}$ is introduced: $\Phi \to \varepsilon\Phi$, $\varepsilon \ll 1$. According to $^{(2)}$, we introduce the operator g

$$F_2(t;p_1,p_2;p_1',p_2') = \gamma_2' F_1(p_1,p_1';t)F_1(p_2,p_2';t) + \varepsilon g(p_1,p_2;p_1',p_2';t),$$

$$F_3(t,p_1,p_2,p_3;p_1',p_2',p_3') - \gamma_3' F_1(p_1,p_1',t)F_1(p_2,p_2',t)F_1(p_3,p_3',t)+$$

$$+\varepsilon\gamma_{13}' g(p_1,p_2;p_1',p_2';t)F_1(p_3,p_3'), \tag{4}$$

where $\gamma_s = \sum_{(P)}(-1)^P$ denotes the antisymmetrized sum over all $s!$ permutations of primed indices. This method makes it possible to obtain an equation for F_1 with no divergent terms in the second order. To obtain a closed equation for F_1 on the basis of approximation (4), it is necessary to find the functional dependence of the correlation operator g on F_1. Substituting (4) into equations (1) for F_1 and F_2, written in the momentum representation, one can show that g satisfies the equation

$$i\hbar\frac{\partial g(p_1,p_2;p_1',p_2',t)}{\partial t} - (T_{p_1} + T_{p_2} - T_{p_1'} - T_{p_2'})g(p_1p_2,p_1'p_2',t)+$$

$$+A(p_1,p_2;p_1',p_2';F_1(t)), \tag{5}$$

$$T_{p_i} = \frac{p_i^2}{2m}.$$

$$\begin{aligned} A(p_1,p_2;p_1',p_2';F_1(t)) = \int dp_1''\,dp_2'' \Big\{ & L_1(p_1,p_2;p_1'',p_2'';p_1'p_2')+ \\ & + \frac{1}{v}\int dp_3\,dp_3''\,L_2(p_1,p_2,p_3;p_1'',p_2'',p_3'';p_1',p_2',p_3)\Big\}, \\ L_1 = {} & \tilde\Phi(p_1,p_2;p_1'',p_2'')\gamma_2'' F_1(p_1'',p_1')F_1(p_2'',p_2')- \\ & - \tilde\Phi(p_1'',p_2'';p_1',p_2')\gamma_2' F_1(p_1,p_1'')F_1(p_2,p_2''), \end{aligned} \tag{6}$$

$$\begin{aligned} L_2 = {} & \{\tilde\Phi(p_1,p_3;p_1'',p_3'')\delta(p_2-p_2'') + \tilde\Phi(p_2,p_3;p_2'',p_3'')\delta(p_1-p_1'')\}\times \\ & \times \gamma_3'' F_1(p_1'',p_1')F_1(p_2'',p_2')F_1(p_3'',p_3)- \\ & - \{\tilde\Phi(p_1'',p_3'';p_1',p_3)\delta(p_2'-p_2'') + \tilde\Phi(p_2'',p_3'';p_2',p_3)\delta(p_1'-p_1'')\}\gamma_3'\times \\ & \times F_1(p_1,p_1'')F_1(p_2,p_2'')F_1(p_3,p_3''). \end{aligned}$$

Let us impose on the solution of equation (5) the boundary condition of weakening of correlations:

$$g(t) \to 0 \quad \text{as} \quad t \to -\infty, \tag{7}$$

used in work (^{4}). This condition proves to be productive in the present work. The solution of equation (5), taking into account the boundary condition (7), has the form:

$$g(p_1, p_2; p_1', p_2', t) = \tag{8}$$

$$= \int_{-\infty}^{t} d\tau \, A(p_1, p_2; p_1', p_2'; \tau) \exp\left\{ \frac{i}{\hbar} (T(p_1) + T(p_2) - T(p_1') - T(p_2'))(t - \tau) \right\}.$$

Taking into account the approximation (4) and substituting the solution found for g into the equation for F_1, we obtain the kinetic equation for a system of fermions with nonlocal interaction in the general spatially inhomogeneous case:

$$\begin{aligned} i\hbar \frac{\partial F_1(t, p_1, p_1')}{\partial t} = {} & (T_{p_1} - T_{p_1'}) F_1(t, p_1, p_1') + \\ & + \frac{\varepsilon}{v} \int dp_2 \, dp_1'' \, dp_2'' \, L_1(p_1, p_2; p_1'', p_2''; p_1', p_2, F_1(t)) + \\ & + \frac{\varepsilon^2}{v} \int dp_2 \, dp_1'' \, dp_2' \{ \tilde{\Phi}(p_1, p_2; p_1'', p_2'') g(p_1'', p_2''; p_1', p_2) - \\ & - \tilde{\Phi}(p_1'', p_2''; p_1', p_2) g(p_1, p_2; p_1'', p_2'') \}, \end{aligned} \tag{9}$$

where g is given by formula (8).

2. Consider small deviations from the equilibrium spatially homogeneous distribution of the system of Fermi particles:

$$F_1(t, p_1, p_1') = F_0(p_1, p_1') + \delta F_1(t, p_1, p_1'), \tag{10}$$

where $F_0(p_1, p_1') = v n(p_1) \delta(p_1 - p_1')$ is a certain equilibrium function of the spatially homogeneous distribution, and $n(p_1)$ are the occupation numbers of states with given momenta and spins.

Substituting (10) into (9) and neglecting terms containing, as a factor, a quantity of second order of smallness $(\delta F_1)^2$, we obtain a variational linearized equation for the small deviation δF_1. Represent δF_1 in the form

$$\delta F_1(tp_1p_1') = f_q(p_1,t)\delta(p_1 - p_1' - q). \tag{11}$$

It can be shown that $f_q(p_1,t)$ satisfies the integral equation:

$$\begin{gathered}
i\hbar\frac{\partial f_q(p_1t)}{\partial t} = (T_{p_1} - T_{p_1-q})f_q(p_1,t)+ \\
+\varepsilon\int dp_2\{R(p_1,p_2)[B(p_1p_2,p_1p_2) + B(p_1-q,p_2,p_1-q,p_2)]+ \\
+B(p_1p_2,p_1-q,p_2+q)[R(p_2,p_1)+R(p_2,p_1-q)]\}+ \\
+\varepsilon^2\int dp_1'\int_{-\infty}^{t} d\tau\int dp_2\,dp_1''\,dp_2''\{\widetilde{\Phi}(p_1p_2,p_1''p_2'')\times \qquad (12) \\
\times\exp\left(\frac{i}{\hbar}(T_{p_1''}+T_{p_2''}-T_{p_1'}-T_{p_2})'(t-\tau)\right)\delta A(p_1''p_2'',p_1',p_2,\tau)- \\
-\widetilde{\Phi}(p_1''p_2'',p_1'p_2)\exp\left(\frac{i}{\hbar}(T_{p_1}+T_{p_2}-T_{p_1''}-T_{p_2''})(t-\tau)\right)\cdot\delta A(p_1p_2,p_1''p_2'',\tau),
\end{gathered}$$

where the following notation has been introduced:

$$R(p_1,p_2) = f_q(q_1)n(p_2), \qquad Q(p_1,p_2p_3) = f_q(p_1)n(p_2)n(p_3),$$

$$B(p_1p_2,p_1'p_2') = \Phi(p_1p_2,p_1'p_2') - \Phi(p_1p_2,p_2'p_1').$$

The varied term $\delta A(p_1p_2,p_1'p_2')$ represents the following sum:

$$\begin{gathered}
\delta A(p_1p_2,p_1'p_2') = A_1(p_1p_2,p_1'p_2')(p_1+p_2-p_1'-p_2'-q)+ \\
+G_1(p_1,p_2)\left[\delta(p_2-p_2')\delta(p_1-p_1'-q)-\delta(p_2-p_1')\delta(p_1-p_2'-q)\right]+ \\
+G_2(p_1,p_2)\left[\delta(p_1-p_1')\delta(p_2-p_2'-q)-(p_1-p_2')\delta(p_2-p_1'-q)\right]; \qquad (13)
\end{gathered}$$

$$\begin{gathered}
A_1(p_1p_2,p_1'p_2') = B(p_1p_2,p_1'+q,p_2')\alpha_1(p_1p_2,p_1'p_2')f_q(p_1'+q)+ \\
+B(p_1p_2,p_1',p_2'+q)\alpha_2(p_1p_2,p_1'p_2')f_q(p_2'+q)- \\
-B(p_1-q,p_2,p_1'p_2')\beta_1(p_1p_2,p_1'p_2')f_q(p_1)-B(p_1,p_2-q,p_1'p_2')\beta_2(p_1p_2,p_1'p_2')f_q(p_2),
\end{gathered}$$

$$\begin{gathered}
\alpha_1(p_1p_2,p_1'p_2') = \alpha_2(p_2p_1,p_2'p_1') = -\beta_1(p_1'p_2',p_1p_2) = -\beta_2(p_2'p_1',p_2p_1) = \\
= n_{p_2'} + n_{p_1}n_{p_2} - n_{p_2}n_{p_2'} - n_{p_1}n_{p_2'}; \qquad (14)
\end{gathered}$$

$$
\begin{gathered}
G_1(p_1, p_2) = G_2(p_2, p_1) = \\
= \frac{1}{v} \int \Big(Q(p_1, p_2 p_3) \{ B(p_1 p_3, p_1 p_3) + B(p_1 - q, p_3, p_1 - q, p_3) + 2B(p_2 p_3, p_2 p_3) \} + \\
+ B(p_1 p_3, p_1 - q, p_3) \{ Q(p_3, p_2, p_1 - q) + Q(p_3 + q, p_2, p_1) \} \Big) \, dp_3. \qquad (15)
\end{gathered}
$$

In what follows we shall consider the case where the momentum is sufficiently small in the sense that the terms of the kinetic equation proportional to $\varepsilon^2 q$ are of higher order of smallness than the terms proportional to ε^2. Since the kinetic equation is derived in the approximation quadratic in ε, in δA one may put $q = 0$, which corresponds to taking into account the leading term of the expansion of δA in powers of q.

In the approximation under consideration, A_1 takes the form

$$
A_1(p_1, p_2; p_1', p_2') = B(p_1, p_2; p_1', p_2') \, S(p_1, p_2; p_1', p_2'; f_q), \qquad (16)
$$

where

$$
\begin{gathered}
S(p_1, p_2; p_1', p_2'; f_q) = \alpha_1(p_1, p_2; p_1', p_2') f_q(p_1') + \\
+ \alpha_2(p_1, p_2; p_1', p_2') f_q(p_2') + \beta_1(p_1, p_2; p_1', p_2') f_q(p_1) + \beta_2(p_1, p_2; p_1', p_2') f_q(p_2).
\end{gathered}
$$

If one takes into account that the coefficient satisfies the symmetry property

$$
S(p_1, p_2; p_1', p_2') = S(p_1, p_2; p_2', p_1') = S(p_2, p_1; p_1', p_2') = -S(p_1', p_2'; p_1, p_2),
$$

then the quadratic term of equation (12) can be symmetrized by means of the mutual interchange of variables $p_1' \leftrightarrow p_2'$, and in the special case of slow change change of $F_1(t)$ in time in comparison with $F_2(t)$ takes the form of the varied collision term

$$
\begin{gathered}
\varepsilon^2 I_B(f_q) = 2\pi\hbar\varepsilon^2 \int dp_2 \, dp_1'' \, dp_2'' \, |B(p_1, p_2; p_1'', p_2'')|^2 \, S(p_1'', p_2''; p_1, p_2) \times \\
\times \, \delta(p_1 + p_2 - p_1'' - p_2'') \delta(T_{p_1} + T_{p_2} - T_{p_1''} - T_{p_2''}), \qquad (17)
\end{gathered}
$$

where $|B(p_1, p_2; p_1'', p_2'')|^2$ is the transition probability between states with different momenta, and

$$
S(p_1'', p_2''; p_1, p_2) = \delta\{ n_{p_1''} n_{p_2''} (1 - n_{p_1})(1 - n_{p_2}) - n_{p_1} n_{p_2} (1 - n_{p_1''})(1 - n_{p_2''}) \}.
$$

The functions $n(p_1 - q)$, $B(p_1 - q, p_2, p_2, p_1 - q)$, which enter the self-consistent-field term, may, for sufficiently small q, be approximately replaced by the first two terms of their expansions in power series in a neighborhood of $q = 0$. Neglecting terms proportional to εq^2, the equation is considerably simplified. In the particular case of a diagonal potential

$$\widetilde{\Phi}(p_1, p_2; p_1', p_2') = \Phi(p_1 - p_2')\delta(p_1 + p_2 - p_1' - p_2')$$

we obtain, without taking into account the term $\varepsilon^2 I_B$, the equation with the self-consistent field (5,6):

$$i\hbar \frac{\partial f_q(p_1, t)}{\partial t} = \left\{ \frac{p_1 q}{m} - \varepsilon q \int dp_2 \, \frac{\partial n_{p_2}}{\partial p_2} \Phi(p_1 - p_2) \right\} f_q(p_1, t) + \varepsilon q \frac{\partial n_{p_1}}{\partial p_1} \int dp_2 f_q(p_2, t) \times$$

$$\times \left(\Phi(p_1 - p_2) - \Phi(q) \right) + \varepsilon^2 I_B(f_q). \tag{18}$$

If, in deriving the kinetic equation, one restricts oneself to terms linear in ε, then after the replacement $p_1 \to p_1 + q/2$ the equation of work (6) can be obtained from (12) without the assumption that q is small.

To find the zeroth approximation it is necessary to take into account the term caused by pair correlation (7), which is responsible for establishing the equilibrium distribution. We note that a nonequilibrium spatially homogeneous distribution

$$F_0(t, p_1, p_1') = vn(p_1, t)\delta(p_1 - p_1')$$

may be chosen as the zeroth approximation; as is seen from (12), it satisfies the equation derived in (4).

From equation (18) one finds (5) the dispersion equation for the zero-sound spectrum, containing, besides the "self-consistent" damping, the damping γ_{st} caused by collisions of particles. For small q, the "self-consistent" damping may be neglected (8).

Taking the frequency of the spectrum to be much greater than the collision frequency $1/\tau$ (τ is the mean free time of the particles) and taking into account that $I_B \sim \varepsilon^2$, we obtain:

$$\gamma_{st} = \varepsilon^2 \int dp_2 \, dp_1'' \, dp_2'' \left| B(p_1, p_2; p_1'', p_2'') \right|^2 \alpha_1(p_1'', p_2''; p_1, p_2) \times$$

$$\times \, \delta(T_{p_1} + T_{p_2} - T_{p_1''} - T_{p_2''}) \delta(p_1 + p_2 - p_1'' - p_2'').$$

The vanishing of this expression at absolute zero temperature does not mean the absence of damping, since, as noted above, the expansion in powers of q of functions that undergo a discontinuity at the Fermi surface is not mathematically justified at $T = 0$.

The authors express their sincere gratitude to Academician N. N. Bogolyubov for his constant attention to the work and for useful advice.

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Received
18 VII 1962

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