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Abstract

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MATHEMATICS

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A VARIATIONAL METHOD FOR SOLVING EXTREMAL PROBLEMS IN CERTAIN CLASSES OF ANALYTIC FUNCTIONS

(Presented by Academician M. A. Lavrent´ ev on 25 III 1963)

The purpose of this note is to extend the Schiffer–Goluzin variational method ([1,2]) to classes of analytic functions (see § 1) whose properties were previously studied by other methods, above all with the aid of structural formulas. For rather general extremal problems we have obtained qualitative characteristics of extremal functions, sometimes sufficient for finding, up to parameters, all extremal functions in such problems.

§ 1. Let U be the class of functions $z = f(w)$, regular in $W: \ |w| < 1$, $f(w) \neq 0$, $f(0) = 0$, taking values only from the disk $|z| < 1$; K the class of functions $z = f(w)$, regular in W, $f(0) \neq 1$, $f'(0) = 1$, having positive real part in W; T_r the class of typically real functions in W, i.e., the aggregate of functions $z = f(w)$, regular in W, $f(0) = 0$, $f'(0) = 1$, for which $\operatorname{Im} f(w)$ and $\operatorname{Im} w$ have the same sign in W if $\operatorname{Im} w \neq 0$; $S^*(\alpha)$ the class of α-spirally starlike functions, i.e., the aggregate of univalent functions $z = f(w)$, regular in W, $f(0) = 0$, $f'(0) = 1$, mapping W onto domains $f(W)$, every point of which can be joined with the origin by an arc of one of the logarithmic spirals of the family

$$z = \exp\left\{-\frac{C}{\sin\alpha} + \frac{\omega}{\operatorname{tg}\alpha} + i\omega\right\}, \qquad -\frac{\pi}{2} < \alpha < \frac{\pi}{2},$$

depending on the real parameter C, under the condition that such an arc lies entirely in $f(W)$; $S^0(\alpha)$ the class of α-spirally convex functions $z = f(w)$, i.e., those for which $wf'(w) \in S^*(\alpha)$, $f(0) = 0$; $V(\alpha)$ the class of functions with bounded rotation, i.e., the set of functions $z = f(w)$, regular in W, $f(0) = 0$, $f'(0) = 1$, for which in W

$$-\frac{\pi}{2} + \alpha < \arg f'(w) < \frac{\pi}{2} + \alpha, \qquad -\frac{\pi}{2} < \alpha < \frac{\pi}{2}.$$

For $\alpha = 0$ the class $S^*(\alpha)$ coincides with the class S^* of starlike functions, and $S^0(\alpha)$ with the class S^0 of convex functions. It is not hard to prove that $V(\alpha)$ is a class of univalent functions.

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Theorem 1. *Suppose that* a_k $(k = 1, 2, \dots, n;\ n = 1, 2, \dots)$ *are arbitrarily fixed points in* W, *and* A_k *are arbitrary complex constants. Let* $f(w)$ *belong to one of the classes* $U,\ K,\ T_r,\ S^*(\alpha),\ S^0(\alpha),\ V(\alpha)$. *Then the function*

$$f_*(w) = f(w) + t\sum_{k=1}^{n} P\left[w;\, a_k, H(a_k), A_k\right] + o(t), \tag{1}$$

$$P\left[w;\, a, H(a), A\right] =$$

$$= AK(w,a)H(a) - \bar{A}K\left(w, \frac{1}{\bar{a}}\right)\overline{H(a)} - AL(w,a) - \bar{A}L\left(w, \frac{1}{\bar{a}}\right), \tag{2}$$

where $t > 0$ and $o(t)$ is a quantity of higher order of smallness than t, on any closed set from W belongs to the same class (i.e., respectively to $U, K, T_r, S^*(a), S^0(a), V(a)$), if $K(\omega, a), L(\omega, a), H(a)$ are defined by the formulas, different for each class, given below.

Namely:

if $f(\omega) \in U$, then

$$K(\omega, a) = \omega[1 - f(\omega)]^2 \frac{\partial}{\partial\omega}\left[\frac{\omega + a}{\omega - a} + H(\omega)\right],$$

$$L(\omega, a) = \omega[1 - f(\omega)]^2 \frac{\partial}{\partial\omega}\left[H(\omega)\frac{\omega + a}{\omega - a}\right],$$

$$H(\omega) = \frac{f(\omega) - 1}{f(\omega) + 1};$$

if $f(\omega) \in K$, then

$$K(\omega, a) = \omega\frac{\partial}{\partial\omega}\left[\frac{\omega + a}{\omega - a} + H(\omega)\right], \qquad L(\omega, a) = \omega\frac{\partial}{\partial\omega}\left[H(\omega)\frac{\omega + a}{\omega - a}\right],$$

$$H(\omega) = f(\omega);$$

if $f(\omega) \in T_r$, then

$$K(\omega, a) = -\frac{\omega^2}{1 - \omega^2}\frac{\partial}{\partial\omega}\left[\frac{\omega(1 - a)^2}{(\omega - a)(1 - a\omega)}\right],$$

$$L(\omega, a) = \frac{\omega^2}{1 - \omega^2}\frac{\partial}{\partial\omega}\left[H(\omega)\frac{a(1 - \omega^2)}{(\omega - a)(1 - a\omega)}\right],$$

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$$H(\omega) = \frac{1-\omega^2}{\omega} f(\omega);$$

if $f(\omega) \in S^*(a)$, then

$$K(\omega, a) = e^{i\alpha} f(\omega) \left[\frac{\omega + a}{\omega - a} + H(\omega) + 1 - e^{-i\alpha} \right],$$

$$L(\omega, a) = e^{i\alpha} f(\omega) \left[H(\omega) \frac{\omega + a}{\omega - a} + e^{-i\alpha} \right], \qquad (*)$$

$$H(\omega) = e^{-i\alpha} \frac{\omega f'(\omega)}{f(\omega)}; \qquad (**)$$

if $f(\omega) \in S^0(a)$, then

$$K(\omega, a) = e^{i\alpha} \int_0^\omega f'(\omega) \left[\frac{\omega + a}{\omega - a} + H(\omega) + 1 - e^{-i\alpha} \right] d\omega,$$

$$L(\omega, a) = e^{i\alpha} \int_0^\omega f'(\omega) \left[H(\omega) \frac{\omega + a}{\omega - a} + e^{-i\alpha} \right] d\omega,$$

$$H(\omega) = e^{-i\alpha} \frac{[\omega f'(\omega)]'}{f'(\omega)};$$

if $f(\omega) \in V(a)$, then

$$K(\omega, a) = e^{i\alpha} \omega \left[\frac{\omega + a}{\omega - a} + H(\omega) \right] - e^{i\alpha} \int_0^\omega \left[\frac{\omega + a}{\omega - a} + H(\omega) \right] d\omega,$$

$$L(\omega, a) = e^{i\alpha} H(\omega) \frac{\omega + a}{\omega - a} - e^{i\alpha} \int_0^\omega H(\omega) \frac{\omega + a}{\omega - a} d\omega, \qquad H(\omega) = e^{-i\alpha} f'(\omega).$$

Let us note that, by choosing in a definite way the arbitrary constants A_k, the number and the position of the points a_k in W, one can obtain variational formulas for subclasses of the classes of functions indicated in Theorem 1.

From the variational formula for $S^*(a)$, if one sets $a = 0$ in it, there follows the variational formula established by us earlier ([3]) in the class of starlike functions.

§ 2. Denote by $I(f)$ a bounded functional defined on some one of the classes of functions under consideration, for example, for definiteness, on the class $S^*(\alpha)$. Suppose that for any function $f(w) \in S^*(\alpha)$, $I(f)$ has a functional derivative

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$$\Phi(f,P)=\lim_{t\to 0}\frac{I(f_*)-I(f)}{t},\qquad t>0,$$

where $f_*(w)=f(w)+tP(w)+o(t)$ is a function of the class $S^*(\alpha)$ close to $f(w)$ (in the sense of uniform convergence inside W), and that $\Phi(f,P)$ is linear with respect to P. We shall say that a functional $I(f)$ possessing the indicated properties is weakly differentiable on $S^*(\alpha)$. Among weakly differentiable functionals are many functionals of interest from the point of view of the theory of functions of a complex variable, in particular the following two functionals of very general form:

$$I(f)=J[f(w),f'(w),\dots,f^{(m)}(w)],\tag{3}$$

$$I(f)=\int_\delta J[f(w),f'(w),\dots,f^{(m)}(w)]\,dS_w.\tag{4}$$

Here f is a function of the class $S^*(\alpha)$, $J(t)=J(t_0,t_1,\dots,t_m)$ is an analytic function of $m+1$ variables $(m=0,1,\dots)$ in some polycylinder E^*. In (3) w is a fixed point in W; in (4) w is a variable from the set δ over which the integration is carried out.

The functional derivative for (3) and (4), respectively, has the form

$$\Phi(f,P)=\sum_{k=0}^{m}\frac{\partial J(f)}{\partial t_k}\,P^{(k)}(w),\qquad \Phi(f,P)=\sum_{k=0}^{m}\int_\delta\frac{\partial J(f)}{\partial t_k}\,P^{(k)}(w)\,dS_w.$$

The set D of values of the weakly differentiable functional $I(f)$ on the class $S^*(\alpha)$ is connected and closed. For each $I_e\in D$ there is an $I_0\in D$ such that

$$|I_0-I_e|=\min_{I\in D}|I-I_e|.$$

A function $f_0(w)\in S^*(\alpha)$ for which $I(f_0)=I_0$, following N. A. Lebedev ([4]), we shall call a boundary function of the functional $I(f)$ on the class $S^*(\alpha)$. Since the closure of the set of points I_0 obtained as I_e varies in CD coincides with the boundary of the domain D, in order to characterize D it is sufficient to find the boundary functions and the points in D produced by them.

Theorem 2. *The boundary functions $f(w)$ of a functional $I(f)$ weakly differentiable on the class $S^*(\alpha)$, with functional derivative $\Phi(f,P)$, satisfy the functional equation*

$$\left\{x\Phi[f,K(w,a)]-\bar{x}\Phi\left[f,K\left(w,\frac{1}{\bar{a}}\right)\right]\right\}H(a)=$$

$$= x\Phi[f, L(w,a)] + \bar{x}\Phi\left[f, L\left(w, \frac{1}{\bar{a}}\right)\right],$$

in which a is an arbitrary point in W, $x = e^{-i\beta}$, $\beta = \arg[I(f) - I_e]$, $I_e \in D$, and $K(w,a)$, $L(w,a)$, $H(a)$ are determined by the formulas $(*)$, $(**)$.

Completely analogous theorems hold for the classes $S^0(\alpha)$, $V(\alpha)$, T_r. Since the classes K and U are not compact, Theorem 2, when applied to them, extends only to those extremal problems whose solutions belong respectively to K and U.

* For example, for (3) $E : \{|t_k| < M_k(w)\ (k = 0, 1, \dots, m)\}, \quad M_k(w) = \sup\limits_{f \in S^*(\alpha)} |f^{(k)}(w)|;$

for (4) $E : \{|t_k| < N_k\ (k = 0, 1, \dots, m)\}, \quad N_k = \sup\limits_{w \in \delta} M_k(w).$

§ 3. **Theorem 3.** Let $I(f)$ be a weakly differentiable functional on the class $S^*(a)$, and suppose that for it $\Phi[f, K(w,a)]$ is, in W, except for a finite set of points, a function of a, analytically continuable to $|w| = 1$. Suppose that

$$\operatorname{Im}\{x\Phi[f, K(w, e^{\varphi i})]\} \not\equiv 0, \qquad 0 \le \varphi < 2\pi.$$

Then each boundary function $z = f(w)$ of the functional $I(f)$ maps W onto a domain $f(W)$ with boundary belonging to the curves determined by the equation

$$\operatorname{Re} H(e^{i\varphi}) = 0.$$

In this theorem $K(w,a)$ and $H(a)$ must be taken from $(*)$ and $(**)$.

Corollary. For the functionals indicated in Theorem 3, in particular for (3) and (4), the boundary functions are

$$f(w) = \eta_0 \exp\left\{ e^{i\alpha} \ln \frac{w}{\prod\limits_{k=1}^{N} (1 - \eta_k w)^{\mu_k}} \right\},$$

where η_k $(k = 0, 1, \dots, N)$ are certain points on $|\eta| = 1$,

$$\sum_{k=1}^{N} \mu_k = 2,$$

and for (3) $N \le 2(m+1)$.

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Theorem 3, with natural modifications, carries over to the classes $S^0(a)$, $V(a)$, T_r, K, and U, which makes it possible also for these classes to give a qualitative characterization of boundary functions in extremal problems. We give two examples.

Theorem 4. The boundary functions $z = f(w)$ of the functionals (3) and (4) on the class T_r map W onto a Riemann surface with edge lying over the real axis of the z-plane.

Theorem 5. If functionals of the form (3) and (4), considered on the class U, are such that the class U contains their extremal functions $f(w)$, then

$$f(w) = \eta w, \qquad |\eta| = 1.$$

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