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Abstract

Full Text

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ON THE EXTENSION OF A MAPPING OF A BOOLEAN RING INTO A TOPOLOGICAL ABELIAN GROUP

(Presented by Academician S. L. Sobolev on 19 XI 1962)

1. Let R be an ordered Boolean ring ($x \leq y \leftrightarrow xy = y$), A a subring of the ring R , and B a complete separated Abelian group. Consider: a set I of indices, a subset \mathcal{A} of the set of all mappings of I into A , and a mapping a of the set \mathcal{A} into R ; a subset \mathcal{B} of the set of all mappings of I into B , and a mapping b of the set \mathcal{B} into B . We shall assume that the sets \mathcal{A} and \mathcal{B} contain the constant 0, and that $a(0) = 0$ and $b(0) = 0$.

Consider a mapping φ of the ring A into the group B such that $\varphi(0) = 0$ and, for every mapping $\xi \in \mathcal{A}$, the superposition $\varphi \circ \xi \in \mathcal{B}$. Denote by \mathfrak{B} the filter of neighborhoods of zero in the group B . We shall suppose additionally that the mapping φ has the following properties:

(P₁) Whatever $V \in \mathfrak{B}$ may be, there exists $V_0 \in \mathfrak{B}$ such that, if $\xi \in \mathcal{A}$, $\eta \in \mathcal{A}$, and $b(\varphi \circ \xi) \in V_0$, $b(\varphi \circ \eta) \in V_0$, then $a(\xi) \leq a(\zeta)$, $a(\eta) \leq a(\zeta)$, and $b(\varphi \circ \zeta) \in V$ for some $\zeta \in \mathcal{A}$.

(P₂) Whatever $V \in \mathfrak{B}$ may be, there exists $V_0 \in \mathfrak{B}$ such that, if $x \in A$, $\xi \in \mathcal{A}$, and $x \leq a(\xi)$, $b(\varphi \circ \xi) \in V_0$, then $\varphi(x) \in V$.

(P₃) Whatever $V \in \mathfrak{B}$ may be, there exists $V_0 \in \mathfrak{B}$ such that, if $x \in A$, $y \in A$, and $\varphi(x + y) \in V_0$, then $\varphi(x) - \varphi(y) \in V$.

To each $V \in \mathfrak{B}$ let us assign $U \subset R$, defined by the property: whatever $x \in U$ may be, there exists $\xi \in \mathcal{A}$ such that $x \leq a(\xi)$ and $b(\varphi \circ \xi) \in V$. The set of all such U is a base of a certain filter \mathfrak{U} in R .

Theorem 1. *There exists a unique topology \mathfrak{T} in R , compatible with the ring structure in R and having \mathfrak{U} as the filter of neighborhoods of zero.*

For the proof of the theorem it suffices to show, using the definition of the filter \mathfrak{U} and property (P₁) of the mapping φ , that \mathfrak{U} satisfies the system of conditions characterizing the filter of neighborhoods of zero in a topological ring ((¹), Ch. 3, §5, item 1).

Consider the topological ring R obtained by endowing the ring R with the topology \mathfrak{T} . From properties (P₂) and (P₃) of the mapping φ it follows directly that

Theorem 2. *The mapping φ is uniformly continuous on A .*

Consider the subring \bar{A} of the topological ring R that is the closure of A . From Theorem 2 and Theorem 1, item 4, § 3, Ch. 2 ⁽¹⁾, it follows directly that

Theorem 3. *There exists a unique mapping $\bar{\varphi}$ of the ring \bar{A} into the group B , uniformly continuous on \bar{A} and extending the mapping φ .*

2. Let us apply the extension scheme just considered to numerical and Boolean measures.
 - a) Let R be the Boolean ring of all subsets of a set E , with the operations of symmetric difference and intersection as addition and multiplication respectively; let A be a subring of the ring R , and let B be the field of real numbers. Consider: the set N of natural numbers, the set \mathcal{A} of all finite (countable) sequences (x_n) of pairwise disjoint elements.

from A and the function $a((x_n)) = \bigcup x_n$, the set \mathcal{B} of all summable sequences (y_n) of elements of B and the function $b((y_n)) = \sum y_n$. We have: $0 \in A$, $0 \in \mathcal{B}$ and $a(0) = 0$, $b(0) = 0$. Consider a measure φ on A that is additive in the finite (countable) sense. We have: $\varphi(0) = 0$ and $(\varphi(x_n)) \in \mathcal{B}$ for $(x_n) \in \mathcal{A}$. From the general properties of a measure (⁽²⁾, § 35 and, respectively, § 37 for the countable case) it follows that φ has properties (P₁)–(P₃). Thus all the conditions of item 1 are fulfilled, and φ can be extended according to the constructed scheme.

The topology \mathcal{T} in the ring R is defined by the fundamental system of neighborhoods $U(\varepsilon)$ of zero (ε is an arbitrary positive number): for any $x \in U(\varepsilon)$, there exists $y \in A$ ($(x_n) \in \mathcal{A}$) such that $x \leq y$ ($x \leq \bigcup x_n$) and $\varphi(x) \leq \varepsilon$ ($\sum \varphi(x_n) \leq \varepsilon$). The closure \bar{A} of the ring A coincides with the set of all subsets of the set E measurable in the sense of Jordan (Lebesgue). The extension $\bar{\varphi}$ of the measure φ coincides with its Jordan (Lebesgue) extension.

- b) Let R and A be the same as in item a). As B , consider the Boolean ring of all subsets of the set F , endowed with the topology of order convergence (⁽³⁾, Chap. IV, § 8). B is a compact totally disconnected ring (⁽⁴⁾, item 4). The fundamental system of neighborhoods of zero in the ring B is the set of all intervals $[0, \varepsilon]$, where ε is the complement of an arbitrary finite subset of the set F . Let, further, $N, \mathcal{A}, \mathcal{B}$ and a, b retain their formal meaning (the terms “summable sequence” and the symbol \sum occurring in the definition of \mathcal{B} are understood in accordance with Definition 1, item 1, § 4, Chap. 3 ⁽¹⁾).

We shall call a finitely additive Boolean measure on A any homomorphism of the ring A into the ring B (cf. ⁽⁵⁾). The notions of countable additivity, measurability in the sense of Jordan (Lebesgue), and Jordan (Lebesgue) extension for Boolean measures are defined analogously to the case of numerical measures (see Definition 1, §§ 36, 37, 39 and Definition 4, § 38 ⁽²⁾, taking into account the new meaning of the symbols \sum and ε).

Consider a Boolean measure φ on A , additive in the finite (countable) sense. We

have $\varphi(0) = 0$. If $(x_n) \in \mathcal{A}$, then $(\varphi(x_n)) \in \mathcal{B}$, since every sequence of disjoint elements of \mathcal{B} is summable. It can also be shown that φ has properties $(P_1) - (P_3)$. Thus all the conditions of item 1 are fulfilled, and φ can be extended according to the constructed scheme.

The assertions concerning \mathcal{J} , \overline{A} , and $\overline{\varphi}$ made in item a) remain valid also for the case under consideration, if the symbols and notions appearing there are used in accordance with the new definitions.

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Note: Figure translations are in progress. See original paper for figures.

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