ON THE NONEXISTENCE OF REGULARLY VARYING TESTS FOR THE BEHRENS–FISHER PROBLEM

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Abstract

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ON THE NONEXISTENCE OF REGULARLY VARYING TESTS FOR THE BEHRENS-FISHER PROBLEM

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The concept of a regularly varying test was introduced in ([1]). We shall consider the question of the existence of similar tests of this type for the Behrens–Fisher problem, treating the latter somewhat broadly.

Let the column vector L_1 and the column vector L_2 be drawn independently, respectively, from n_1- and n_2-dimensional normal populations with column vectors of means Λ_1, Λ_2 and matrices of second moments $\sigma_1^2 E_{n_1 n_1}, \sigma_2^2 E_{n_2 n_2}$. Let $\Lambda_1 = A_1 \Xi_1$ and $\Lambda_2 = A_2 \Xi_2$, where $A_i = (A_i)_{n_i m_i}$, $\text{rank}(A_i) = m_i < n_i$, $\Xi_i = (\Xi_i)_{m_i,1}$, $i = 1, 2$. Introduce matrices $G_i = (G_i)_{km_i}$ of rank $k \leq m_i$, $i = 1, 2$, and construct the column vector $H = G_1 \Xi_1 + G_2 \Xi_2$. The vectors Ξ_1 and Ξ_2 will be regarded as unknown, as will σ_1, σ_2 and the ratio $\vartheta = \sigma_1^2 / \sigma_2^2$. Under these conditions we are interested in tests suitable for testing the hypothesis $H = 0$ (0 is the null vector) and satisfying certain restrictions.

We shall require that such a test for our problem be defined by an appropriately measurable function $G(L_1, L_2)$, for which the critical regions $\mathfrak{R}_C : G(L_1, L_2) \geq C$ would satisfy the following conditions:

1. The regions $\mathfrak{R}_C$ lie in the space of sufficient statistics. The likelihood function for L_1 and L_2 can be represented in the form

$$\frac{1}{(2\pi)^{\frac{n_1+n_2}{2}} \sigma_1^{n_1} \sigma_2^{n_2}} \exp\left\{ -\frac{1}{2} \sum_{i=1}^{2} \frac{1}{\sigma_i^2} \left([vv]_i + X_i^T B_{iX} i - 2\Xi_i^T B_{iX} i + \Xi_i^T B_i \Xi_i \right) \right\},$$

where $B_i = A_i^T A_i$, $X_i = B_i^{-1} A_i^T L_i$, $[vv]_i = (L_i - A_{iX} i)^T (L_i - A_{iX} i)$. This gives the sufficient statistics $X_1, X_2, [vv]_1, [vv]_2$. Condition 1 means that if $(L_1, L_2) \in \mathfrak{R}_C$, and for (L_1', L_2') $X_i' = X_i$, $[vv]_i' = [vv]_i$, $i = 1, 2$, then also $(L_1', L_2') \in \mathfrak{R}_C$.

2. If $(L_1, L_2) \in \mathfrak{R}_C$, then $(L_1 + A_1 C_1, L_2 + A_2 C_2) \in \mathfrak{R}_C$ for any vectors $C_i = (C_i)_{m_i,1}$ for which $G_1 C_1 + G_2 C_2 = 0$.

3. If $(L_1, L_2) \in \mathfrak{R}_C$, then for any $k \neq 0$, $(kL_1, kL_2) \in \mathfrak{R}_C$.

Let us note that these conditions generalize in a natural way the well-known axioms of A. Wald ([2]). Moreover, by a method analogous to that given in ([3]), one can construct "approximate" similar regions of any size; these regions will satisfy the three conditions formulated.

Conditions 1-3 determine the form of the function G. Applying condition 2 first with $C_1 = -X_1 + G_1^T(G_1G_1^T)^{-1}G_1X_1$ and $C_2 = 0$, and then with $C_1 = G_1^T(G_1G_1^T)^{-1}G_2X_2$ and $C_2 = -X_2$; applying condition 3 with $k = 1/\sqrt{[vv]_2}$, we find $G = g(H, h)$, where $H = (G_1X_1 + G_2X_2)/\sqrt{[vv]_2}$, $h = \sqrt{[vv]_1/[vv]_2}$, and $(H, h) \in \Omega = (-\infty < H < \infty,\ 0 \le h < \infty)$. In the space Ω we have a one-parameter family of densities

$$C\vartheta^{\frac{n_2-m_2+k}{2}}|M|^{\frac{n_1-m_1+n_2-m_2+k-1}{2}}\frac{h^{n_1-m_1-1}}{[\,|M|(\vartheta H^TM^{-1}H + h^2 + \vartheta)\,]^{\frac{n_1-m_1+n_2-m_2+k}{2}}},$$

$$M = \vartheta G_1B_1^{-1}G_1^T + G_2B_2^{-1}G_2^T.$$

The denominator

$$|M|(\vartheta H^TM^{-1}H + h^2 + \vartheta)$$

is a polynomial of degree $k+1$; all its roots are real and negative. For the root whose absolute value is greater than the maximal eigenvalue of the regular pencil of forms

$$G_2B_2^{-1}G_2^T - \lambda G_1B_1^{-1}G_1^T,$$

we always have

$$|\vartheta| > H^T(G_1B_1^{-1}G_1^T)^{-1}H + h^2.$$

By the boundary measure, at least one such root exists. Let $\vartheta_i = \vartheta_i(H, h)$, $i = 1, \dots, k+1$, be the roots, with $|\vartheta_{k+1}|$ greater than the indicated maximal eigenvalue.

Consider a regularly varying similar test $g(H, h)$. For definiteness, let

$$\operatorname{vrai}\max_K g(H,h) < \operatorname{vrai}\max_\Omega g(H,h)$$

whatever half-ball $K \subset \Omega$ with center at the origin is taken. By choosing the radius of the half-ball large, one can push the root ϑ_{k+1} arbitrarily far away. Define the function

$$\Psi(z) = \begin{cases} 1, & z > \operatorname{vrai}\max_K g(H,h), \\ 0, & z < \operatorname{vrai}\max_K g(H,h). \end{cases}$$

From the condition of similarity it follows that

$$\int_\Omega \Psi[g(H,h)] \frac{h^{n_1-m_1-1}\, dH\, dh}{[(\vartheta-\vartheta_1)\cdots(\vartheta-\vartheta_{k+1})]^{\frac{n_1-m_1+n_2-m_2+k}{2}}} =$$

$$= C_\Psi\, \vartheta^{-\frac{n_2-m_2+k}{2}} \left[\prod_{i=1}^{k}(\vartheta+\lambda_i)\right]^{-\frac{n_1-m_1+n_2-m_2+k-1}{2}}, \qquad (1)$$

where $\lambda_1, \dots, \lambda_k$ are the eigenvalues of the pencil

$$G_2 B_2^{-1} G_2^T - \lambda G_1 B_1^{-1} G_1^T.$$

The integral relation (1) admits analytic continuation in ϑ, in any case into a small angle containing the positive half-axis. Meanwhile, after moving slightly from a sufficiently distant point of the half-axis, one can find that the right- and left-hand sides of (1) cannot coincide. Let $\vartheta = Re^{i\varphi}$, with R sufficiently large and $\varphi > 0$ sufficiently small. Then the values of the integrand in (1) (and, consequently, the integral over Ω itself) will lie inside a certain angle, whereas the right-hand side of (1) will lie outside this angle. This follows from the inequality

$$(n_1-m_1+n_2-m_2+k)\sum_{i=1}^{k+1}\arg(\vartheta-\vartheta_i) \le$$

$$\le \varepsilon + (n_1-m_1+n_2-m_2+k)\,k\,\arg\vartheta < (n_2-m_2+k)\arg\vartheta +$$

$$+(n_1-m_1+n_2+m_2+k-1)\,k\,\arg\left(\vartheta+\max_i\lambda_i\right),$$

which holds for a large radius of the half-ball K, large R, and small $\varphi > 0$.

Thus, the following assertion is true:

Theorem. *A regularly varying similar test for the Behrens-Fisher problem cannot exist.*

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CITED LITERATURE

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2. A. Wald, *Selected Papers in Statistics and Probability*, N. Y., 1955, p. 669.
3. O. V. Shalaevskii, DAN, **130**, No. 1 (1960).

Note: Figure translations are in progress. See original paper for figures.