ON THE CONTINUITY OF THE RADON INTEGRAL AS A FUNCTION OF A PARAMETER

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Abstract

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MATHEMATICS

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ON THE CONTINUITY OF THE RADON INTEGRAL AS A FUNCTION OF A PARAMETER

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Let $X = \{x\}$ be a set of arbitrary nature and let $\mathfrak{M} = \{e\}$ be a family of subsets of the set X, containing all of X, as well as the differences and finite or countable sums of its elements. Further, let α be a parameter whose values belong to some metric space. We shall consider measures $M_\alpha(e)$, depending on α, defined on $\mathfrak{M}$, i.e., nonnegative and completely additive functions of $e \in \mathfrak{M}$, and also functions $f_\alpha(x)$, measurable with respect to $\mathfrak{M}$ and summable with respect to $M_\alpha(e)$ (in both cases α is the same!). Then the Radon integrals

$$F(\alpha) = \int_X f_\alpha(x) M_\alpha(de). \tag{1}$$

make sense.

V. M. Dubrovsky ([1]) found a condition for the continuity of the function $F(\alpha)$, consisting (besides the continuity in α, for fixed x and e, of the functions $f_\alpha(x)$ and $M_\alpha(e)$) in the uniform with respect to α convergence of the integral (1). By this is meant the uniform with respect to α tendency to zero (as $n \to +\infty$) of the difference

$$\int_X |f_\alpha(x)|\, M_\alpha(de) - \int_X \left[\, |f_\alpha(x)|\, \right]_n M_\alpha(de), \tag{2}$$

where, as usual, $[f(x)]_n = \min\{f(x), n\}$ is the "cut-off" of the nonnegative function $f(x)$ by the number n.

There, too, V. M. Dubrovsky establishes a sufficient (but not necessary!) criterion for the uniform with respect to α convergence of the integral (1), consisting in the inequality

$$\int_X |f_\alpha(x)|^p M_\alpha(de) < K,$$

where $p > 1$ and K do not depend on α.

In the present work a necessary and sufficient criterion is established for the uniform with respect to α convergence of the integral (1), reminiscent of the well-known Vallée-Poussin criterion for the equi-absolute continuity of the Lebesgue integral. This analogy is natural, since it is not difficult to show that the equi-absolute continuity of the integral

$$(L)\int_a^b f_\alpha(x)\,dx \tag{3}$$

is equivalent to its uniform with respect to α convergence in the sense of V. M. Dubrovsky.

Theorem 1. *Suppose there exists an increasing function $\Phi(u) \geq 0$, defined for $0 \leq u < +\infty$, continuous at $u = +\infty$, with $\Phi(+\infty) = +\infty$, such that for all α one has*

$$\int_X |f_\alpha(x)|\,\Phi[\,|f_\alpha(x)|\,]M_\alpha(de) < K, \tag{4}$$

where K is a finite constant independent of α. Then the integral (1) converges uniformly with respect to α.

Proof. We shall first of all show that the integral (1) is uniformly absolutely continuous. By this we mean the following: to every $\varepsilon > 0$ there corresponds a $\delta = \delta(\varepsilon)$ such that the inequality $M_\alpha(E) < \delta$, where $E \in \mathfrak{M}$, implies the inequality

$$\int_E |f_\alpha(x)|\,M_\alpha(de) < \varepsilon. \tag{5}$$

Indeed, let $E \in \mathfrak{M}$. Choosing any natural number n and some α, put

$$\underline{E} = E_x(|f_\alpha(x)| \leq n), \qquad \overline{E} = E_x(|f_\alpha(x)| > n).$$

Then

$$\int_E |f_\alpha(x)|\,M_\alpha(de) = \int_{\underline{E}} + \int_{\overline{E}} \leq nM_\alpha(\underline{E}) + \int_{\overline{E}} |f_\alpha(x)|\frac{\Phi[|f_\alpha(x)|]}{\Phi(n)}\,M_\alpha(de) \leq$$

$$\leq nM_\alpha(E) + \frac{K}{\Phi(n)}.$$

Having noted this, fix such an n that $K/\Phi(n) < \varepsilon/2$. Then the desired value of $\delta(\varepsilon)$ will be the number $\varepsilon/2n$.

Turning now to the theorem itself, suppose that it is false. Then there exist $\varepsilon_0 > 0$, a sequence $n_k \uparrow \infty$, and a sequence α_k for which

$$\int_X |f_{\alpha_k}(x)|\, M_{\alpha_k}(de) - \int_X [|f_{\alpha_k}(x)|]_{n_k}\, M_{\alpha_k}(de) \geq \varepsilon_0.$$

Put $E_k = X(|f_{\alpha_k}(x)| > n_k)$. Then

$$\int_{E_k} |f_{\alpha_k}(x)|\, M_{\alpha_k}(de) \geq \varepsilon_0 + n_k M_{\alpha_k}(E_k)$$

and, a fortiori,

$$\int_{E_k} |f_{\alpha_k}(x)|\, M_{\alpha_k}(de) \geq \varepsilon_0,$$

whence

$$M_{\alpha_k}(E_k) \geq \delta(\varepsilon_0). \tag{6}$$

On the other hand,

$$K \geq \int_X |f_{\alpha_k}(x)|\, \Phi[|f_{\alpha_k}(x)|]\, M_{\alpha_k}(de) \geq \int_{E_k} n_k \Phi(n_k)\, M_{\alpha_k}(de)$$

and, consequently,

$$M_{\alpha_k}(E_k) \leq \frac{K}{n_k \Phi(n_k)},$$

which contradicts inequality (6). The theorem is proved.

Lemma 1. *If the integral (1) converges uniformly with respect to α, then it is uniformly absolutely continuous.*

Take $\varepsilon > 0$ and fix such an n that, for all α, the difference (2) is $< \varepsilon/2$. Then for any $E \in \mathfrak{M}$ we will have, a fortiori,

$$\int_E |f_\alpha(x)|\, M_\alpha(de) < \frac{\varepsilon}{2} + n M_\alpha(E).$$

Consequently, the inequality $M_\alpha(E) < \varepsilon/2n$ ensures estimate (5).

Lemma 2. *Under the conditions of Lemma 1, uniformly with respect to α we have*

$$\lim_{Q\to+\infty} M_\alpha[X(|f_\alpha| \geq Q)] = 0. \tag{7}$$

If the lemma is false, then for some $\delta_0 > 0$ there exist a sequence $Q_k \to +\infty$ and a sequence α_k such that

$$M_{\alpha_k}[X(|f_{\alpha_k}| \geq Q_k)] > \delta_0. \tag{8}$$

Noting this, let us fix such an n that for all α

$$\int_X |f_\alpha(x)|\, M_\alpha(de) - \int_X [|f_\alpha(x)|]_n\, M_\alpha(de) < 1.$$

Then, putting for brevity $X[|f_{\alpha_k}| \geq Q_k] = E_k$, we shall have

$$\int_{E_k} |f_{\alpha_k}(x)|\, M_{\alpha_k}(de) - \int_{E_k} [|f_{\alpha_k}(x)|]_n\, M_{\alpha_k}(de) < 1.$$

Thus

$$\int_{E_k} |f_{\alpha_k}(x)|\, M_{\alpha_k}(de) < 1 + nM_{\alpha_k}(E_k).$$

On the other hand, the last integral is not less than $Q_k M_{\alpha_k}(E_k)$, whence $M_{\alpha_k}(E_k) > 1/(Q_k - n)$, which for sufficiently large k is incompatible with (8).

Theorem 2. If integral (1) converges uniformly with respect to α, then there exists a function $\Phi(u)$ possessing the properties described in Theorem 1 and satisfying relation (4) for all α.

Proof. Let $0 < Q_1 < Q_2 < \cdots \to +\infty$ and, for all α,

$$M_\alpha[X(|f_\alpha| \geq Q_k)] > \delta\left(\frac{1}{k^3}\right).$$

The existence of such Q_k follows from Lemma 2. Put $Q_0 = 0$, and let $\Phi(u) = k$ for $Q_k \leq u \leq Q_{k+1}$. It is clear that the $\Phi(u)$ constructed by us has the properties described in Theorem 1. On the other hand, putting $X_k = X(Q_k \leq |f_\alpha| < Q_{k+1})$, we have

$$\int_X |f_\alpha(x)|\, \Phi[|f_\alpha(x)|]\, M_\alpha(de) = \sum_{k=0}^{\infty} \int_{X_k} = \sum_{k=0}^{\infty} k \int_{X_k} |f_\alpha(x)|\, M_\alpha(de) <$$

$$< \sum_{k=1}^{\infty} k \frac{1}{k^3} = \frac{\pi^2}{6}.$$

The theorem is proved.

Remark. If the Lebesgue integral (3) converges uniformly with respect to α, then it is bounded. However, under the more general conditions of Theorem 1, integral (1) may also be unbounded. For example, let $X = [0, 1]$, let $\mathfrak{M}$ be the family of L-measurable subsets of X, and let $M_\alpha(e) = \alpha me$ $(0 < \alpha < +\infty)$, while $f_\alpha(x) \equiv 1$. Then integral (1) converges uniformly with respect to α, but it is equal to α.

Let us also note that in the situation under consideration, uniform convergence with respect to α of integral (1) no longer follows from its equi-absolutely continuity, understood in the sense introduced above. For example, let $X = \{x_k\}$ be a countable set, let α_k take all possible positive rational values,

$$M_{\alpha_k}(e) = \begin{cases} \alpha_k, & \text{if } x_k \in e, \\ 0, & \text{if } x_k \notin e; \end{cases} \qquad f_{\alpha_k}(x) = \begin{cases} \alpha_k, & \text{if } x = x_k, \\ 0, & \text{if } x \neq x_k. \end{cases}$$

Then

$$\int_E f_{\alpha_k}(x)\, M_{\alpha_k}(de) = \begin{cases} \alpha_k^2, & \text{if } x_k \in E, \\ 0, & \text{if } x_k \notin E, \end{cases}$$

and, consequently, integral (1) is equi-absolutely continuous. At the same time, convergence (7) is not uniform with respect to α_k.

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CITED LITERATURE

1. V. M. Dubrovsky, DAN, **66**, No. 2, 149 (1949).

Note: Figure translations are in progress. See original paper for figures.