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Abstract

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ON SOME BOUNDARY-VALUE PROBLEMS FOR THE EQUATION

$$u_{xx} + \operatorname{sign} y\, u_{yy} = 0$$

(Presented by Academician M. A. Lavrent' ev on 22 XII 1962)

In the present paper two boundary-value problems are considered for the Lavrent' ev–Bitsadze equation

$$u_{xx} + \operatorname{sign} y\, u_{yy} = 0 \tag{1}$$

of type of problem T_1 (1–3).

Let D be a simply connected domain of the plane xy, bounded by a Jordan curve σ with endpoints at $A(-1,0)$, $B(1,0)$, situated in the upper half-plane $y>0$, and by the characteristics $AC:\ y=-x-1$ and $BC:\ y=x-1$, issuing from the point $C(0,-1)$. Let $E_k(a_k,0)$, $k=1,\dots,n$, $-1<a_1<\dots<a_n<1$, be prescribed points of the segment AB. The points $A_k[\,\frac{1}{2}(a_k-1), -\frac{1}{2}(a_k+1)\,]$ and $B_k[\,\frac{1}{2}(a_k+1), \frac{1}{2}(a_k-1)\,]$, $k=0,1,\dots,n+1$ ($a_0=-1,\ a_{n+1}=1$), lie respectively on the characteristics AC and BC. Denote by $E_{ik}[\,\frac{1}{2}(a_i+a_k), \frac{1}{2}(a_i-a_k)\,]$, $i\le k,\ i=0,1,\dots,n;\ k=1,\dots,n+1$, the point of intersection of the characteristics E_iB_i and E_kA_k ($E_0=A,\ E_{n+1}=B,\ E_{0k}=A_k,\ E_{k,n+1}=B_k$). Denote by D_1 and D_2, respectively, the elliptic and hyperbolic parts of the mixed domain D.

Problem T_1^1. It is required to determine a function $u(x,y)$ with the following properties: 1) $u(x,y)$ is a solution of equation (1) in the domain D everywhere except for the points of the segment AB, the real axis, and the characteristics E_kA_k, E_kB_k; 2) $u(x,y)$ is continuous in the closed domain $\overline{D}$; 3) the partial derivatives u_x and u_y are continuously matched at all points of the segment AB, except, possibly, for the points $E_k,\ k=0,1,\dots,n+1$, at which u_x and u_y may tend to infinity of order less than one; 4) $u(x,y)$ assumes the prescribed values

$$u=\varphi \quad \text{on } \sigma; \tag{2}$$

$$u=\psi_k \text{ on } E_kE_{k-1},\ k \text{ odd}; \qquad u=\psi_k \text{ on } E_{k-1}E_{k-1},\ k \text{ even}, \tag{3}$$

where φ is continuous, while $\psi_k(x)$, $k = 1, \dots, n+1$, are twice differentiable functions whose second derivatives satisfy a Hölder condition, and moreover $\psi_{2k-1}(a_{2k-1}) = \psi_{2k}(a_{2k-1})$, $k = 1, 2, \dots$ (for $n = 2m$ the condition $\psi_{n+1}(1) = \varphi(1)$ must also be satisfied).

Problem T_1^1 in the case $n = 1$, $a_1 = 0$ was investigated in the work of T. D. Dzhuraev ([4])*. A problem of type T_1^1 in the case $n = 1$ was first posed and investigated in the work of Gellerstedt ([5]) for the equation $y^m z_{xx} + z_{yy} = 0$.

Let $n = 2m$. The case $n = 2m-1$ is investigated analogously. In the domain D_2 the solution $u(x, y)$ of equation (1) has the form

$$u(x,y) = \frac{\tau(x+y) + \tau(x-y)}{2} + \frac{1}{2}\int_{x-y}^{x+y} \nu(t)\,dt, \tag{4}$$

* Arbitrary constants c_1 and c_2 cannot be determined by the method proposed in ([4]). It is easy to see that the function $F(z)$ satisfies the condition $F(1/\overline{z}) = -F(z)$, whence it follows that $\operatorname{Re} F(z) = 0$ for $z = -1$ and $z = 1$ for any c_1 and c_2. These constants may be determined from the conditions $F(-1) = 0$ and $\operatorname{Re} F(0) = \psi_1(0) - \omega_2(0) - W(0,0)$.

where $\tau(x) = u(x, 0)$, $-1 \le x \le 1$, and $\nu(x) = u_y(x, 0)$, $-1 < x < 1$.

By virtue of (3), from (4) we obtain

$$u_x - \lambda(x) u_y = f(x), \quad y = 0, \quad a_k < x < a_{k+1}, \quad k = 0, 1, \dots, 2m, \tag{5}$$

where $\lambda(x) = -1$ on L_1; $\lambda(x) = 1$ on L_2; $f(x) = \psi'_{2k-1}\left[\frac{1}{2}(x + a_{2k-1})\right]$ on L_1; $f(x) = \psi'_{2k}\left[\frac{1}{2}(x + a_{2k-1})\right]$ on L_2; L_1 and L_2 are, respectively, the union of the intervals (a_{2k-2}, a_{2k-1}), $k = 1, \dots, m+1$, and (a_{2k-1}, a_{2k}), $k = 1, \dots, m$.

Hence, analogously to problem T_1 for $n = 2m-1$, we conclude that if $\psi_k(x) \equiv 0$, $k = 1, \dots, 2m+1$, then the solution $u(x, y)$ of problem T_1^1 in the closed domain $\overline{D}_1$ attains a nonzero extremum on the arc σ (the extremum principle). From this principle the uniqueness of the solution of problem T_1^1 follows immediately.

Without loss of generality one may assume that $\varphi \equiv 0$ ([6]). We shall additionally assume that σ is a smooth arc satisfying Lyapunov' s condition, and that u_x and u_y are continuous in the closed domain $\overline{D}_1$ everywhere except, possibly, at the points E_k, $k = 0, 1, \dots, 2m+1$. By a conformal mapping one can arrange that σ coincide with the semicircle σ_0 with endpoints at the points A and B ([6]). We shall assume that σ coincides with σ_0.

Denote by $\Phi(z)$ the function $u(x, y) + iv(x, y)$, holomorphic in the domain D_1 and satisfying the condition $\Phi(-1) = 0$.

The conditions (5) may be written in the form

$$\text{Re}(1-i)\Phi'(x) = f(x) \text{ on } L_2, \qquad \text{Im}(1-i)\Phi'(x) = -f(x) \text{ on } L_1. \tag{6}$$

By virtue of the condition $u = 0$ on σ_0, we conclude that the function $\Phi(z)$ is analytically continued through σ_0 to the entire upper half-plane, and

$$\Phi(z) = \begin{cases} u(x,y) + iv(x,y), & \text{inside } D_1, \\ -u\left[\dfrac{x}{x^2+y^2}, \dfrac{y}{x^2+y^2}\right] + i\,v\left[\dfrac{x}{x^2+y^2}, \dfrac{y}{x^2+y^2}\right], & \text{outside } D_1. \end{cases} \tag{7}$$

Hence it follows that the function $\Phi(z)$ must satisfy the condition

$$\overline{\Phi(1/\bar{z})} = -\Phi(z). \tag{8}$$

At infinity $\Phi'(z)$ has a zero of second order owing to the boundedness of $u(x,y)$ (7).

Let $a_{2j-1} < 0 < a_{2j}$. From (7) and (6) we obtain

$$\begin{aligned} \text{Re}(1-i)\Phi'(x) &= (1/x^2)\, f(1/x) \qquad \text{on } \overline{L}_1, \\ \text{Im}(1-i)\Phi'(x) &= -(1/x^2)\, f(1/x) \qquad \text{on } \overline{L}_2, \end{aligned} \tag{9}$$

where $\overline{L}_1$ and $\overline{L}_2$ denote, respectively, the union of the intervals (b_{2k-1}, b_{2k-2}), $k = 1, \dots, m+1$, and (b_{2k}, b_{2k-1}), $k = 1, \dots, j-1, j+1, \dots, m$, $(-\infty, b_{2j-1})$, (b_{2j}, ∞), and $b_k = 1/a_k$.

Thus, the determination of the function $\Phi'(z)$ is reduced to determining, in the upper half-plane, a piecewise holomorphic function $\Phi'(z)$ having a zero of second order at infinity and satisfying the boundary conditions (6) and (9).

The solution of this problem of class h_0 is given by the well-known Keldysh-Sedov formula ([7],[8])

$$(1-i)\Phi'(z) = \frac{1}{\pi i} \frac{R_1(z)}{R_2(z)} \int_{-\infty}^{\infty} \frac{R_2(t)}{R_1(t)} \frac{g(t)}{t-z}\, dt + \frac{C_0 + C_1 z + \dots + C_{2m-1} z^{2m-1}}{R(z)}, \tag{10}$$

where $g(x) = f(x)$ on L_2; $g(x) = -if(x)$ on L_1; $x^2 g(x) = f(1/x)$ on $\overline{L}_1$;

$$x^2 g(x) = -if(1/x) \quad \text{on } \overline{L}_2 \quad \text{and} \quad R_1(z) = \left[(z-1)\prod_1^m (z-a_{2k-1})(z-b_{2k-1})\right]^{1/2};$$

$$R_2(z) = \left[(z+1)\prod_1^m (z-a_{2k})(z-b_{2k})\right]^{1/2}, \quad R(z) = \left[(z^2-1)\prod_1^{2m}(z-a_k)(z-b_k)\right]^{1/2},$$

where by $R_1(z)/R_2(z)$ we mean the branch holomorphic in the plane cut along $L_2, \overline{L}_1$, taking the value 1 at infinity, and by $R(z)$ the branch holomorphic in the plane cut in the same way, taking positive values on Ox for $x > b_{2j}$; $C_0, C_1, \dots, C_{2m-1}$ are arbitrary real constants.

After determining $\Phi'(z)$, the function $\Phi(z)$ is found from the formula

$$\Phi(z) = \int_{-1}^{z} \Phi'(\zeta)\, d\zeta. \tag{11}$$

It is easy to see that, for condition (8) to hold, it is necessary and sufficient that

$$C_k = -C_{2m-k-1}, \quad k = 0, 1, \dots, m-1.$$

To determine $C_k,\ k = 0, 1, \dots, m-1$, we have the following conditions:

$$\operatorname{Re}\Phi(a_{2k-1}) = \psi_{2k-1}(a_{2k-1}), \quad k = 1, \dots, m. \tag{12}$$

These conditions constitute a system of m linear equations with respect to $C_k,\ k = 0, 1, \dots, m-1$:

$$\sum_{j=0}^{m-1} \gamma_{kj} C_j = \gamma_k, \quad k = 1, \dots, m, \tag{13}$$

where γ_{kj} do not depend on $\psi_k(x)$, while $\gamma_k = 0$ when $\psi_k(x) \equiv 0$.

From the uniqueness of the solution of problem T_1^1 it follows directly that system (13) is uniquely solvable.

The real part of the function $\Phi'(z)$ gives the required function $u(x, y)$ in the domain D_1. In the domain D_2, the solution $u(x, y)$ is given by formula (4), where $\tau(x)$ and $\nu(x)$ are determined respectively from (11) and (10).

Remark. To prove the existence of a solution one may use the method of integral equations. Just as in problem T_1 (^{2}), to determine the function $\nu(x)$ (it is assumed that σ coincides with σ_0 and $\varphi = 0$) one obtains the singular integral equation

$$\lambda(x)\nu(x) + \frac{1}{\pi}\int_{-1}^{1}\left(\frac{1}{t-x} - \frac{t}{1-tx}\right)\nu(t)\, dt = -f(x). \tag{14}$$

Analogously to paper [9], we conclude that the solution* of this equation, belonging to the class h_0 and satisfying the Hölder condition, is given by the formula

$$\nu(x) = -\operatorname{Im}\Phi'^{+}(x),$$

where $\Phi'^{+}(x)$ is determined from (10).

Problem T_0. It is required to determine a function $u(x, y)$ with the following properties: 1) $u(x, y)$ is a solution of equation (1) in the domain D everywhere except at the points of the segment AB and the characteristics $E_kA_k,\ E_kB_k$; 2) $u(x, y)$ is continuous in the closed domain $\overline{D}$; 3) the partial derivatives u_x and u_y are continuously matched at all points of the segment AB, except, possibly, the points $E_k,\ k = 1, \dots, n$, at which u_x and u_y may tend to infinity of logarithmic type, and the points A, B, at which u_x and u_y may tend to infin–

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* In paper [2] only one particular solution of integral equation (11), belonging to the class of sought solutions, was obtained. With its help it is impossible to construct a solution of problem T_1 for $n > 1$. This is obvious if $\psi_k = \alpha_k,\ k = 0, 1, \dots, 2m$, where α_k are constants, and $\varphi = 0$.

of finiteness of order less than unity; 4) $u(x, y)$ assumes the prescribed values

$$u = \varphi \text{ on } \sigma, \qquad u = \psi_k(x) \text{ on } A_kA_{k+1}, \quad k = 0, 1, \dots, n, \tag{15}$$

where φ is continuous, and the $\psi_k(x)$ are twice differentiable functions whose second derivatives satisfy Hölder' s condition, with

$$\varphi(-1) = \psi_0(-1), \quad \psi_k\left[\frac{1}{2}(a_{k+1} - 1)\right] = \psi_{k+1}\left[\frac{1}{2}(a_{k+1} - 1)\right], \quad k = 0, 1, \dots, n{-}1.$$

This boundary-value problem is a generalization of the Tricomi problem [1,3] in the case when the first derivative of the boundary function in the domain D_2 has a finite number of discontinuities of the first kind.

In essence, T_0 is a problem of the type of problem T_1^1 and is investigated analogously to it. For the function $\Phi'(z)$ we obtain

$$\Phi'(z) = \frac{1-i}{2\pi}\left(\frac{z+1}{z-1}\right)^{1/2}\int_{-1}^{1}\left(\frac{t-1}{t+1}\right)^{1/2}\left(\frac{1}{t-z} - \frac{t}{1-tz}\right) f(t)\, dt,$$

where

$$f(x) = \psi_k'\left[\frac{1}{2}(x-1)\right], \qquad a_k < x < a_{k+1}, \quad k = 0, 1, \dots, n,$$

and by $[(z+1)/(z-1)]^{1/2}$ is meant the branch holomorphic in the plane cut along $(-1,1)$ and taking the value 1 at infinity.

The function $\Phi'(z)$ is bounded at $z=-1$, becomes infinite of order 1/2 at $z=1$, and has a logarithmic singularity at the points $a_k,\ k=1,\dots,n$.

In an analogous way one can generalize problem T_1^1.

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Note: Figure translations are in progress. See original paper for figures.