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Abstract

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ON AN ASYMPTOTIC METHOD IN THE THEORY OF PLASMA STABILITY

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In finding the eigenvalues of the frequencies ω in the linear theory of the stability of an inhomogeneous rarefied plasma in a strong magnetic field $H^{(0)}$, there arises a generalized eigenvalue problem for an integro-differential operator of the form:

$$\hat{L}(x,\omega)\psi(x) = \int_{-\infty}^{+\infty} d\xi \int_a^b d\tau\, K(\xi,\omega,\tau)\,\delta(\xi - x - \varepsilon f(\tau))\psi(\xi). \qquad (1)$$

Here $\hat{L}(x,\omega)$ is a certain linear differential operator of second order, $\hat{L}(x,\omega) \equiv \varepsilon^2 d^2/dx^2 + a(x,\omega)$, ε is a small parameter, and the coefficient $a(x,\omega)$ and the kernel $K(x,\omega,\tau)$ depend parametrically on ω and are slowly varying functions in the sense that $\varepsilon\, d\ln K(x)/dx,\ \varepsilon\, d\ln a(x)/dx \ll 1$.

For example, in the problem of the so-called "universal instability" of an inhomogeneous plasma, the integral operator $\hat{K}$ has the form:

$$\hat{K}^{i,e}\psi(x) \equiv \int_{-\infty}^{+\infty} d\xi \int_0^{\infty} \exp\left[-\frac{m_{i,e}v_\perp^2}{2T(\xi)}\right] v_\perp dv_\perp \times$$

$$\times \int_0^{2\pi} d\varphi \int_0^{2\pi/\omega_{Hi,e}} dt\, \exp\left\{\frac{ik_\perp v_\perp}{\omega_{Hi,e}}[\cos(\varphi - \omega_{Hi,e}t) - \cos\varphi]\right\}, \qquad (1a)$$

$$\delta\left(\xi - x\frac{v_\perp}{\omega_{Hi,e}}[\sin(\varphi - \omega_{Hi,e}t) - \sin\varphi]\right)\psi(\xi),$$

and the role of the small parameter is played by the Larmor radius

$$r_{i,e} = \sqrt{\frac{T}{m_{i,e}}}\,\omega_{Hi,e}^{-1} \left(\omega_{Hi,e} = \frac{e_{i,e}H^{(0)}}{m_{i,e}c}, \quad \omega \ll \omega_{Hi}\right)$$

($T(x)$ is the temperature of the electrons and ions). In this case the operator corresponding to (1) is complex even for real x. The imaginary part in it arises from taking into account the pole by-pass in the integrals

$$\int_{-\infty}^{+\infty} \frac{f_{i,e}^{(0)}}{\omega + k_\parallel v_\parallel}\, dv_\parallel$$

with the unperturbed distribution function of ions and electrons $f_{i,e}^{(0)}(x, \mathbf{v})$. These pole by-passes describe the interaction of "drift" waves with resonant ions and electrons ($v_\parallel = -\omega/k_\parallel$).

We shall further assume that equation (1) can be considered in the complex plane z, and, in solving the eigenvalue problem for ω, we shall require that the eigenfunctions satisfy the condition $\psi(x) \to 0$ as $x \to \pm\infty$. The presence of the small parameter ε in equation (1) makes it possible to use asymptotic methods analogous to the WKB method in quantum mechanics. We represent $\psi(x)$ in the form:

$$\psi(x) = \exp\left\{ i \int^{x} k_x(x,\omega)\, dx \right\}, \tag{2}$$

where $k_x(x,\omega) = k_x^{(0)}(x,\omega) + \varepsilon k_x^{(1)}(x,\omega) + \varepsilon^2 k_x^{(2)}(x,\omega) + \cdots,\ k_x^{(\alpha)}(x,\omega)$ are slowly varying functions $\left(d \ln k_x^{(\alpha)}/dx \ll 1\right)$.

In the first approximation we obtain for $k_x^{(0)}$ a certain, generally speaking, transcendental equation:

$$-\varepsilon^2 \big[k_x^{(0)}(x,\omega)\big]^2 + a(x,\omega) = \int_a^b K(x,\omega,\tau) \exp\big[ik^{(0)}(x,\omega)\varepsilon f(\tau)\big]\, d\tau. \tag{3}$$

For simplicity we shall restrict ourselves to the case in which (3) has the form

$$-\varepsilon^2 \big[k_x^{(0)}(x,\omega)\big]^2 + a(x,\omega) - F(x,\omega,\varepsilon^2[k_x^{(0)}(x,\omega)]^2) = 0 \tag{4}$$

(it is precisely this case that occurs for the operator (1a)). The roots of equation (4) have the form $\pm k_{xj}^{(0)}(x,\omega)$ $(j = 1, 2, ...)$.

If there exists a root such that $\operatorname{Im} k_j^{(0)}(x) \neq 0$ as $x \to \pm\infty$, then the corresponding solution $\psi(x) \to 0$ as $x \to \pm\infty$ in two cases:
a) when $\operatorname{Im} k_j^{(0)}(x,\omega)$ has different signs on the half-axes $x \to -\infty$ and $x \to +\infty$, and we can, without crossing Stokes lines, continue solution (2) from the region $x \to -\infty$ into the region $x \to +\infty$; b) when $\operatorname{Im} k_j^{(0)}(x,\omega)$ has the same sign as $x \to \pm\infty$, but at the so-called turning points z_1, z_2 (where $k_x^{(0)}(z_{1,2},\omega) = 0$ and the WKB approximation is already invalid) there occurs a change of the solutions (2) corresponding to the different signs of the root $\pm k_j^{(0)}$.

In the second case, when continuing the approximate solution (2) from the region $x \to -\infty$ into the region $x \to +\infty$, it is necessary to match it with the exact solution of equation (1) near the turning points. To construct the exact solution, we represent the integral operator in the form of a differential operator of infinite order, which in the case under consideration is the formal expansion in a series in $\varepsilon^2 d^2/dx^2$ of the function $F(x, \omega, -\varepsilon^2 d^2/dx^2)$ (see (4)). Approximating, near the turning point z_1, the small difference $F(z, \omega, 0) - a(z, \omega)$ by two terms of the expansion in a series,

$$F(z, \omega, 0) - a(z, \omega) = -(z - z_1)/R,$$

we obtain the differential equation in dimensionless form

$$\frac{d^2\psi(\tilde{z})}{d\tilde{z}^2} + \tilde{z}\psi - \sum_{l=2}^{\infty}(1-\alpha_2)^{-1}\left(\frac{\varepsilon}{R(1-\alpha_2)}\right)^{\frac{2(l-1)}{3}} \alpha_{2l}\frac{d^{2l}\psi}{d\tilde{z}^{2l}} = 0, \qquad (5)$$

where

$$\alpha_{2l} = \frac{(-1)^l}{l!}\left[\frac{d^l}{d\theta^l}F(z_1, \omega, \theta)\right]_{\theta=0}$$

are coefficients which, near the turning point, may be regarded as constant, $\tilde{z} = (z - z_1)/(R\varepsilon^2(1-\alpha_2))^{1/3}$.

From this equation it is clear that there are only two roots $k_x^{(0)}(\tilde{z}) \simeq \pm\sqrt{-\tilde{z}}$ that can vanish as $\tilde{z} \to 0$. For these two roots we must find the exact solution for $\tilde{z} \gg 1$, where the WKB approximation is already applicable, and consequently the approximate and exact solutions can be matched in the common domain of their applicability. The remaining roots $k_{xj}^{(0)}$, even as $\tilde{z} \to 0$, remain finite, and for them the validity of the WKB approximation is nowhere violated.

Solving equation (5) by means of the generalized Laplace method, it is easy to show that the asymptotic behavior of the solutions corresponding to the roots $k_x^{(0)} \simeq \pm\sqrt{-\tilde{z}}$ is the same as that of the Airy functions

$$\psi(z) = \exp\left\{\pm\frac{2}{3}(-\tilde{z})^{3/2}\right\}. \qquad (6)$$

As is seen from (6), in the complex z-plane three rays L emanate from the turning point z_1, on which the solution is purely oscillatory, and three rays N, on which it is purely real and monotone. When the condition is violated

$|(z - z_1)/R| \ll 1$, these rays pass into the lines of constant real

Fig. 1

Figure 1: Fig. 1

$$\operatorname{Im} \int_{z_1}^{z} k_x(z)\, dz = 0, \qquad z \in L,$$

and imaginary

$$\operatorname{Re} \int_{z_1}^{z} k_x(z)\, dz = 0, \qquad z \in N,$$

phase of the function (2) in the WKB approximation (see Fig. 1). Let us note that if the solution $\psi(z)$ is defined as decaying ($\psi(z) \to 0$) as $|z| \to \infty$, $z \in N_{01}$, then the lines L will bound the region $z \in D_1$, $N_{01} \in D_1$, in which $\psi(z) \to 0$ as $|z| \to 0$. Taking this remark into account, and also the fact that, by virtue of (5), the relation between solutions decaying on the line N_{01} and oscillating on L_0 is the same as in ordinary quantum mechanics, we obtain the condition for determining the eigenfrequencies ω corresponding to solutions $\psi(z) \to 0$, $|z| \to \infty$, $z \in D_{1,2}$:

Fig. 1

$$\int_{z_1}^{z_2} k_x^{(0)}(z, \omega^{(p)})\, dz = \left(p + \frac{1}{2}\right)\pi, \qquad (p = 0, 1, 2, \ldots). \tag{7}$$

in the case when there are two turning points z_1, z_2 connected by the line L_0 (see Fig. 1). (In contrast to ordinary quantum mechanics, the integration here is carried out in the complex z-plane along the line L_0, and the wave number must be determined from the transcendental equation (3).) The solution $\psi(x) \to 0$ in this case as $x \to \pm\infty$, if the regions $D_{1,2}$ include both half-axes $x \to \pm\infty$ (Fig. 1).

Thus, the eigenfrequencies $\omega^{(p)}$ can be obtained from condition (7) if the analytic functions $a(z, \omega)$, $K(z, \omega, \tau)$ are known and $k_x^{(0)}(z, \omega)$ is determined from the transcendental equation (3). However, in stability theory it is often sufficient to restrict oneself to the sign and order of magnitude of the increment $\nu = -\operatorname{Im}\omega$. For finding them, integral relations are useful; these are easily obtained from the differential equation

$$\varepsilon^2 d^2\psi/dx^2 + a(x, \omega)\psi - F(x, \omega, -\varepsilon^2 d^2/dx^2)\psi = 0, \tag{8}$$

corresponding to (4). Indeed, let us multiply it by $\psi^*(x)$ and integrate over "infinite" limits. Then, using the form (2) of the function $\psi(x)$ and integration by parts, we obtain:

$$\varepsilon^2 \int_{-\infty}^{+\infty} \left|\frac{d\psi}{dx}\right|^2 dx - \mathrm{Re} \int_{-\infty}^{+\infty} \left[a(x,\omega) - F(x,\omega,\varepsilon^2|k_x(x,\omega)|^2)\right] |\psi|^2 \, dx = 0,$$

$$\mathrm{Im} \int_{-\infty}^{+\infty} \left[a(x,\omega) - F(x,\omega,\varepsilon^2|k_x(x,\omega)|^2)\right] |\psi|^2 dx = 0. \tag{9}$$

The second condition in plasma stability theory has the simple physical meaning of the law of conservation of energy in the system wave perturbation + plasma particles (as we have already noted, it is precisely the imaginary part of the integrals

$$\int \frac{f_{i,e}^{(0)}}{\omega + k_\parallel v_\parallel} \, dv_\parallel,$$

entering into the expressions for $a(x)$ and $F(x)$, that describes the interaction of the wave and the particles).

In the most frequently occurring case, when $U(x) \equiv -\mathrm{Re}\, k_x^2(x,\omega)$ has the form of a "well" ($U(\pm\infty) > 0$, $\pm U'(x) < 0$ for $x_1 < x < x_1$), and $|\mathrm{Im}\, k_x| \ll |k_x|$, the function $\psi(x)$ is substantially different from zero and is almost purely oscillatory inside the "well," but almost immediately beyond the points x_1, x_2 decays exponentially. In this case it follows from the second condition (9) that

$$\mathrm{Im} \left\{ a\left(x^{(p)}, \omega^{(p)}\right) - F\left(x^{(p)}, \omega^{(p)}, \varepsilon^2 \left|k_x\left(x^{(p)}, \omega^{(p)}\right)\right|^2\right)\right\} = 0, \quad x_1 < x^{(p)} < x_2. \tag{10a}$$

This condition, together with the equation

$$\mathrm{Re}\; \varepsilon^2 k_x^2\left(x^{(p)}, \omega^{(p)}\right) - \mathrm{Re}\left\{a\left(x^{(p)}, \omega^{(p)}\right) - F\left(x^{(p)}, \omega^{(p)}, \varepsilon^2 k_x^2\right)\right\} = 0, \qquad (10\ )$$

which follows from (8), can serve to determine the sign and order of magnitude of $\mathrm{Im}\, \omega^{(p)}$. The equation (1) obtained in the particular problem of the so-called "universal instability" of an inhomogeneous rarefied plasma is given in Ref. (1) and in the general case has a very cumbersome form. Therefore we shall confine ourselves only to presenting the results of solving this problem by means of (10a,).

In the case of parallelism of the magnetic-field lines $H^{(0)}$, in the frequency range $k_z u_i < \omega < k_z V_A$ ($u_i = \sqrt{T/m_i}$, $V_A = \sqrt{H^2/4\pi n_0 m_i}$, n_0, T being the density and temperature of the plasma, the perturbation ψ is chosen in the form $\psi \equiv \psi(x) \times \exp\{i\omega t + ik_y y + ik_z z\}$), in the plasma, even in the absence of a temperature gradient ($dT/dx \equiv 0$), an instability develops with increment

$\nu \sim \dfrac{ck_{yT}}{en_0 H}\dfrac{dn_0}{dx}$ for perturbations with wavelengths $\lambda_x \sim r_i$ (2). If, however, one takes into account the effect of nonparallelism of the magnetic-field lines ("shear"), when their angle of inclination $\theta(x)$ relative to the z-axis (the field $H^{(0)}$ then lies in the (y, z) plane) changes with the coordinate x, then already under the condition*

$$R\frac{d\theta}{dx} \sim \int_{x_1}^{x_2} \frac{d\theta}{dx}\, dx > \frac{r_i}{R}, \tag{11}$$

where R is the characteristic scale on which the density $n(x)$ changes appreciably, the instability in the absence of a temperature gradient $T(x)$ is stabilized. This occurs because in the second condition (9) one can no longer neglect, throughout the entire localization region of the perturbation, the half-taking in the integrals with the ion distribution function

$$\int_{-\infty}^{+\infty} \frac{f_i^{(0)}}{\omega + k_\parallel v_\parallel}\, dv_\parallel \quad \left(k_\parallel(x) = k_z + k_y \int^x \frac{d\theta}{dx}\, dx\right),$$

which describes "Landau damping" on ions.

For "shear"of order (11), "drift"nonpotential waves ($\omega > k_\parallel V_A$) do not interact, in their localization region, with the ions and remain unstable, though only when $d\ln T/d\ln n < 0$. The instability with respect to these perturbations can be stabilized by narrowing the "well" $U(x)$ at such a "shear," when the wavelength λ_x becomes, in order of magnitude, greater than the width $X = x_2 - x_1$ of the "well" and no finite solutions $\psi(x)$ exist. The greatest value of the shear $R\, d\theta/dx > \sqrt{8\pi nT/H^2}$ is required for stabilization of perturbations in the frequency region $\omega > k_z u_e$.

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* This result agrees with the preliminary consideration (3).

Note: Figure translations are in progress. See original paper for figures.