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Abstract

Full Text

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On Stability in Liouville's Theorem on Conformal Mappings of Space

(Presented by Academician S. L. Sobolev on 4 IV 1963)

1. Let E^n be n-dimensional Euclidean space, $|x|$ the length of a vector $x \in E^n$, and Q_r the ball $\{|x| < r\}$, $0 \leq r < 1$. In the space E^n we assume a fixed Cartesian orthonormal coordinate system. Let $f(x)$ be a mapping of a domain $M \subset E^n$ into E^n. We shall say that the mapping $f(x)$ belongs to the class $W_n^1(M)$ if the coordinates $f_1(x), f_2(x), \dots, f_n(x)$ of the vector function $f(x)$ have first derivatives, generalized in the sense of S. L. Sobolev (^{1}), summable to the power n in the domain M. If, in addition, $f(x)$ is a topological mapping of the domain M into itself, then we shall say that $f(x) \in W_{n,T}^1(M)$.

For an arbitrary mapping $f(x) \in W_n^1(M)$ we set:

$$\lambda(x, f) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial x_j} \right)^2, \qquad D(f, M) = \int_M [\lambda(x, f)]^{n/2} \, dx,$$

$$J(x, f) = \frac{\partial(f_1, f_2, \dots, f_n)}{\partial(x_1, x_2, \dots, x_n)} = \det \left\| \frac{\partial f_i}{\partial x_j} \right\|, \qquad V(f, M) = \int_M J(x, f) \, dx.$$

If the mapping $f(x) \in W_{n,T}^1(M)$, then $V(f, M) \neq 0$. We set

$$\theta(f) = \frac{D(f, M)}{n^{n/2} |V(f, M)|}.$$

Theorem 1. There exists a universal function $\alpha(\varepsilon, r) \geq 0$, defined for $\varepsilon > 0$, $0 \leq r < 1$, with $\alpha(\varepsilon, r) \to 0$ as $\varepsilon \to 0$, such that for every mapping $f(x) \in W_{n,T}^1$ of the ball Q_1 into itself for which $\theta(f) \leq 1 + \varepsilon$, there exists a Möbius mapping $g(x)$ such that, for $|x| \leq r < 1$, the inequality

$$|f(x) - g(x)| \leq \alpha(\varepsilon, r)$$

holds.

Remark. The function $\alpha(\varepsilon, r)$ in the theorem just formulated cannot be replaced by a function $\alpha(\varepsilon)$ independent of r.

Theorem 2. If a mapping $f(x) \in W^1_{n,T}$ and $\theta(f) = 1$, then the mapping $f(x)$ is Möbius.

Theorem 1 contains, as a special case, the theorem from the author' s work (^{2}). The proofs of Theorems 1 and 2 are based on considerations analogous to those given in (^{2}).

The equicontinuity of a sequence of mappings of the class $W^1_{n,T}$ of the ball Q_1 into itself is ensured by the following lemma.

Lemma. Let $f(x)$ be a mapping of the class $W^1_n(Q_1)$ of the ball Q_1 such that $D(f, Q_1) \leq D_0 < \infty$. Then for every point $x_0 \in D_1$ and for every

$\delta \in [0, 1]$ there exists an r such that $\delta < r < \sqrt{\delta}$, and the vector-function $f(x)$ is continuous on the sphere $S_r = \{|x - x_0| = r\}$. Moreover, for any $x_1, x_2 \in S_r$ the inequality

$$|f(x_1) - f(x_2)| \leq C_n \left(\frac{D_0}{\ln 1/\delta}\right)^{1/n} \left|\frac{x_1 - x_2}{r}\right|^{1/n},$$

holds, where C_n is a constant.

In the case $n = 2$, this lemma coincides with the well-known lemma of R. Courant (^{3}). The proof of the lemma in the general case is based on considerations analogous to those used when $n = 2$. In doing so it is necessary to apply the embedding theorems of S. L. Sobolev (^{1}).

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CITED LITERATURE

[1] S. L. Sobolev, *Some Applications of Functional Analysis to Mathematical Physics*, 1951.
[2] Yu. G. Reshetnyak, in: *Some Problems of Mathematics and Mechanics*, Novosibirsk, 1961, p. 219.
[3] R. Courant, *Dirichlet' s Principle, Conformal Mappings, and Minimal Surfaces*, IL, 1953.

Note: Figure translations are in progress. See original paper for figures.