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Abstract

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ON THE STRUCTURE OF FACTOR-REPRESENTATIONS OF A LOCALLY COMPACT GROUP

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1. As is known ([1],[2]), a continuous sum of equivalent or quasi-equivalent factor-representations of a locally compact group G is also a factor-representation; in particular, a continuous sum of mutually equivalent irreducible representations is a factor-representation of type I. On the other hand, a continuous sum of pairwise non-quasi-equivalent factor-representations may turn out to be a factor-representation.

Every factor-representation of type I is a discrete sum of mutually equivalent irreducible representations; the question of the structure of factor-representations of types II and III still remains open. In the decomposition of a factor-representation of type II or III there must occur mutually non-equivalent irreducible representations; therefore the question of the structure of these factor-representations is closely connected with the necessity of generalizing the very concept of equivalence of representations. The question of sufficient conditions (more precise than those indicated above) under which a continuous sum of factor-representations is also a factor-representation likewise leads to the need for such a generalization. The sufficient condition set forth below is apparently already close to being necessary and to some extent sheds light on the structure of factor-representations of types II and III.

In order not to complicate the exposition, certain simplifying assumptions are made here. Let us note that the results presented are immediately carried over to symmetric representations of rings with involution.

2. Let G be a locally compact group satisfying the second axiom of countability; ν a right-invariant Haar measure on G; S a space with finite Borel measure μ, and let to each $s \in S$ there be assigned a factor-representation $g \to U_g(s)$ of the group G in one and the same separable Hilbert space H. Denote by

$$\mathfrak{H} = \int_S \oplus H \, d\mu(s)$$

the continuous sum of the spaces $H(s) = H$ with respect to the measure μ, and by $g \to U_g = \{U_g(s)\}$ the corresponding continuous sum of the representations $g \to U_g(s)$ in $\mathfrak{H}$. The representations $g \to U_g(s)$ will be called **weakly equivalent to one another** (relative to (S, μ)) if there exist:

1) a linear set D, dense in H, invariant under all operators $U_g(s)$, $g \in G$, $s \in S$;
2) a linear set $\mathfrak{D}$, dense in $\mathfrak{H}$, of functions $\xi = \{\xi(s)\} \in \mathfrak{H}$ with values in D, invariant under all operators U_g, $g \in G$.
3) a bilinear form $(\xi, \eta)_{s_1, s_2}$, defined for all $\xi, \eta \in D$ and all $s_1, s_2 \in S$;
4) a set F of functions $f(s_1, s_2) \in L^1(S \times S)$, dense in $L^1(S \times S)$, possessing the following properties:

a) $\mathfrak{D}$ contains all functions $\xi(s) = \text{const} = \xi \in D$;

b) $(\xi(s_1), \eta(s_2))_{s_1, s_2}$ is a measurable function on $S_1 \times S_2$, essentially bounded on $S_1 \times S_2$, for any $\xi(s), \eta(s) \in \mathfrak{D}$;

c) there exist $\xi_0, \eta_0 \in D$ such that $(\xi_0, \eta_0)_{s_1, s_2} \neq 0$ for almost all pairs $s_1, s_2 \in S \times S$;

d) $(U_g(s_1)\xi, \eta)_{s_1, s_2} = (\xi, U_{g^{-1}}(s_2)\eta)_{s_1, s_2}$ for almost all $s_1, s_2 \in S \times S$, for arbitrary $\xi, \eta \in D$;

e) if $\xi = \{\xi(s)\}$, $\eta = \{\eta(s)\} \in \mathfrak{D}$ and $f(s_1, s_2) \in F$ is any function, then the formula

$$(\xi, \eta)' = \int f(s_1, s_2) \, (\xi(s_1), \eta(s_2))_{s_1, s_2} \, d\mu(s_1) d\mu(s_2)$$

defines on $\mathfrak{D}$ a bounded bilinear form $(\xi, \eta)'$.

If the form $(\xi, \eta)_{s_1, s_2}$ is bounded, then there exists a bounded operator $A(s_1, s_2) \neq 0$ such that

$$(\xi, \eta)_{s_1, s_2} = (A(s_1, s_2)\xi, \eta),$$

where (ξ, η) is the scalar product in H; condition d) in this case means that

$$A(s_1, s_2) U_g(s_1) = U_g(s_2) A(s_1, s_2),$$

i.e. that the representations $g \to U_g(s_1)$ and $g \to U_g(s_2)$ are quasiequivalent (and, in the case of their irreducibility, equivalent). However, in the definition given above it is not required that the form $(\xi, \eta)_{s_1, s_2}$ be bounded, and the operators $A(s_1, s_2)$ (as operators in the Hilbert space) need not exist at all, just as the value of a generalized function at a fixed point does not exist; according to this definition it makes no sense to speak of the equivalence of an individual pair of representations $g \to U_g(s_1)$, $g \to U_g(s_2)$, but only of the equivalence, among themselves (relative to (S, μ)), of all representations $g \to U_g(s)$, $s \in S$.

Theorem. *A continuous sum of factor representations that are weakly equivalent to one another is also a factor representation.*

Proof. We retain the preceding notation and denote also by $M(s)$ and M the weakly closed rings generated by the operators $U_g(s)$ and U_g, respectively. By assumption, $M(s)$ is a factor, and we must prove that M is also a factor. Let Z be the center of the ring M, and let $\widetilde{M}$ be the set of all operators $A = \{A(s)\}$ in $\mathfrak{H}$, where $A(s)$ is a μ-measurable operator-valued function, essentially bounded in norm, with values $A(s) \in M(s)$. Choose in $L^1(G)$ an everywhere-countable dense set $\{x_n(g)\}$; then the operators

$$U_{x_n} = \int x_n(g) U_g \, d\nu(g)$$

generate M, and

$$U_{x_n}(s) = \int x_n(g) U_g(s) \, d\nu(g)$$

are measurable operator functions of s with values in $M(s)$, generating $M(s)$ for each fixed $s \in S$. This means that the factors $M(s)$ form a measurable family, and therefore $\widetilde{M}$ is weakly closed (see, for example, (^{3}), p. 180). But, obviously, the operators $U_g = \{U_g(s)\} \in \widetilde{M}$; hence the weakly closed ring M generated by them is contained in $\widetilde{M}$, and a fortiori $Z \subset \widetilde{M}$. Let $A \in Z$; then $A \in \widetilde{M}$, and consequently has the form $A = \{A(s)\}$, where $A(s)$ is a μ-measurable essentially norm-bounded operator-valued function with values $A(s) \in M(s)$. But since $A \in Z$, we have $AU_{x_n} = U_{x_n} A$ for all $n = 1, 2, ...$; consequently, for fixed n,

$$A(s) U_{x_n}(s) = U_{x_n}(s) A(s) \tag{1}$$

for $s \in S - S_n$, where $\mu(S_n) = 0$. Put $S' = \bigcup_{n=1}^{\infty} S_n$; then (1) holds for all $n = 1, 2, ...$ when $s \in S - S'$, where $\mu(S') = 0$. Since the operators $U_n(s)$, $n = 1, 2, ...$, generate the factor $M(s)$, it follows from (1) that, for $s \in S - S'$, $A(s)$ belongs to the center of the factor $M(s)$ and therefore is a multiple of the identity operator: $A(s) = \lambda(s) 1$. Consequently, all operators in Z have the form $A = \{\lambda(s) 1\}$, where $\lambda(s) \in L^\infty(S, \mu)$, and it remains

show that in fact these functions $\lambda(s) = \text{const}$. For this we shall use the condition of weak equivalence of the representations $g \to U_g(s)$. Let B be a bounded linear operator in $\mathfrak{H}$, defined by a form $(\xi, \eta)'$ in $\mathfrak{D}$, such that $(\xi, \eta)' = (B\xi, \eta)$, where $(\, , \,)$ is the scalar product in $\mathfrak{H}$. The operator B commutes with all operators U_g; indeed, by condition d), for $\xi = \{\xi(s)\}$, $\eta = \{\eta(s)\} \in \mathfrak{D}$,

$$(BU_g \xi, \eta) = (U_g \xi, \eta)' = \int f(s_1, s_2) (U_g(s_1) \xi(s_1), \eta(s_2))_{s_1, s_2} \, d\mu(s_1) d\mu(s_2) =$$

$$= \int f(s_1, s_2) (\xi(s_1), U_{g^{-1}}(s_2) \eta(s_2))_{s_1, s_2} \, d\mu(s_1) d\mu(s_2) =$$

$$= (\xi, U_{g^{-1}}\eta)' = (B\xi, U_{g^{-1}}\eta) = (U_{gB}\xi, \eta),$$

and since $\mathfrak{D}$ is dense in $\mathfrak{H}$, it follows that $BU_g = U_{gB}$. But then B commutes with all operators from M and, in particular, with all operators $A = \{\lambda(s)1\} \in Z$. Hence $(BA\xi, \eta) = (AB\xi, \eta) = (B\xi, A^*\eta)$, i.e. $(A\xi, \eta)' = (\xi, A^*\eta)'$. This means that

$$\int f(s_1, s_2)(\lambda(s_1)\xi(s_1), \eta(s_2))_{s_1,s_2}\, d\mu(s_1)d\mu(s_2) =$$

$$= \int f(s_1, s_2)(\xi(s_1), \overline{\lambda(s_2)}\eta(s_2))_{s_1,s_2}\, d\mu(s_1)d\mu(s_2);$$

therefore, by conditions (4) and b),

$$(\lambda(s_1)\xi(s_1), \eta(s_2))_{s_1,s_2} = (\xi(s_1), \overline{\lambda(s_2)}\eta(s_2))_{s_1,s_2};$$

i.e.

$$\lambda(s_1)(\xi(s_1), \eta(s_2))_{s_1,s_2} = \lambda(s_2)(\xi(s_1), \eta(s_2))_{s_1,s_2} \tag{2}$$

for almost all pairs s_1, s_2. Using now conditions a) and c), put in (2) $\xi(s) = \text{const} = \xi_0$ and $\eta(s) = \text{const} = \eta_0$; we obtain that

$$\lambda(s_1)(\xi_0, \eta_0)_{s_1,s_2} = \lambda(s_2)(\xi_0, \eta_0)_{s_1,s_2}$$

for almost all pairs $s_1, s_2 \in S \times S$, where $(\xi_0, \eta_0)_{s_1,s_2} \neq 0$. Hence $\lambda(s_1) = \lambda(s_2)$ for almost all pairs s_1, s_2; consequently, $\lambda(s) = \text{const}$ almost everywhere on S. Thus Z consists only of multiples of the identity operator, i.e. M is a factor.

The theorem proved is, obviously, applicable to the case of weakly equivalent irreducible representations, since irreducible representations are also factor representations. If among these irreducible representations there are some inequivalent in the ordinary sense, then their continuous sum must be a factor representation of type II or III.

3. Let us consider, as an example, the discrete group G of all transformations $y = \alpha x + \beta$, where $\alpha \neq 0$ and β are rational numbers. Let $g \to U_g$ be the left regular representation of the group G in the space $L^2(G)$; it is known that this representation is a factor representation.* We shall establish this fact by applying the theorem proved in § 2. To this end we first decompose the representation $g \to U_g$ into irreducible representations. Let B be the additive group of rational numbers β; A, the multiplicative group of rational numbers $\alpha \neq 0$; S, its character group $s = s(\alpha)$ (compact, since

A is discrete); and μ, Haar measure on S. For any function $f = f(\alpha, \beta) \in L^2(G)$ put $\xi_s(\beta) = \sum_\alpha f(\alpha, \beta) s(\alpha)$. Since

$$\|f\|^2 = \sum_{\alpha,\beta} |f(\alpha, \beta)|^2 = \int_S \sum_\beta |\xi_s(\beta)|^2 \, d\mu(s),$$

then the correspondence $f \to \xi_s$ is an isometric mapping of the space $L^2(G)$ onto

$$\mathfrak{H} = \int_S^\oplus L^2(B) \, d\mu(s);$$

as is easily verified, upon applying the operator U_{g_0} to $f(\alpha, \beta)$ the function $\xi_s(\beta)$ passes into

$$U_{g_0}(s)\xi_s(\beta) = \overline{s(\alpha_0)} \, \xi_s(\alpha_0^{-1}(\beta - \beta_0)),$$

so that the representation $g \to U_g$ is a continuous sum of the representations $g \to U_g(s)$. It is also easy to see that the representations $g \to U_g(s)$ are irreducible and pairwise inequivalent in the ordinary sense. We shall show that these representations are weakly equivalent to one another. Let D be the set of all finite (i.e., nonzero only for a finite number of values of β) elements $\xi(\beta)$ of $H = L^2(B)$; F the set of all functions on S which are Fourier transforms of finite functions $f(\alpha)$ on A; $\mathfrak{D}$ the set of all functions $\xi_s(\beta)$ which, for each β, take values from F, and for which the set of values of β independent of s and different from zero is finite; F the set of all finite linear combinations of the functions $f_1(s_1) f_2(s_2)$, where $f_1, f_2 \in \mathfrak{F}$. Put, for $\xi_1, \xi_2 \in D$,

$$(\xi_1, \xi_2)_{s_1, s_2} = \sum_{\beta \neq \beta'} s'(\beta - \beta') \, \overline{s_2(\beta - \beta')} \, \xi_1(\beta) \, \overline{\xi_2(\beta')};$$

it is not hard to verify that, with this, all the conditions in the definition of weak equivalence will be satisfied for the representations $g \to U_g(s)$; consequently, these representations are weakly equivalent to one another, and $g \to U_g$ is a factor-representation on the basis of the theorem of item 2.

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* Moreover of type II (see, for example, (4), pp. 434-437).

REFERENCES

1. F. J. Mautner, Ann. Math., **52**, 528 (1950).
2. M. A. Naimark, Sibirsk. matem. zhurn., **2**, 1, 89 (1961).
3. J. Dixmier, *Les algèbres d' opérateurs dans l' espace hilbertien*, Paris, 1957.
4. M. A. Naimark, *Normed Rings*, Moscow, 1956.

Note: Figure translations are in progress. See original paper for figures.

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