Reports of the Academy of Sciences of the USSR

PHYSICS

1963

SovietRxiv

Abstract

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Reports of the Academy of Sciences of the USSR
1963. Volume 152, No. 3

PHYSICS

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ON THE THEORY OF WEAKLY TURBULENT PLASMA

(Presented by Academician M. A. Leontovich on 6 IV 1963)

In relaxation processes occurring in a collisionless plasma, nonlinear effects play an essential role. Part of these effects is taken into account in the quasilinear theory ($^{1-3}$), where the back reaction of the oscillations on the particle distribution function is considered. In doing so, however, one neglects the nonlinear interaction between waves, which in a number of cases can substantially affect the relaxation of oscillations. The interaction between harmonics was studied earlier in (^{4}) for a plasma in a strong magnetic field ($H^2/8\pi \gg nT$) and in (^{5}) for an arbitrary transparent medium. However, the general method set forth in (^{5}) is also applicable to the case of weak "opacity," when the decrements (increments) of the waves are sufficiently small ($\gamma \ll \omega$), which makes it possible to consider the interaction of oscillations with one another and with the plasma from a unified point of view and to estimate more accurately than was done previously (^{3}) the role of the interaction of oscillations in relaxation processes.*

In what follows, for simplicity of exposition, we shall consider a plasma without a magnetic field. The considerations presented below, however, are also valid in the general case. The corresponding formulas taking the magnetic field into account will be given elsewhere in connection with applications.

Following (^{4,5}), let us write the expression for the electric field of the plasma oscillations in the form

$$\mathbf{E}(\mathbf{r},t) = \sum_{\mathbf{p}=\mathbf{k},\,\mathbf{k}_-} \mathbf{E}_{\mathbf{p}}(t) e^{i(\mathbf{pr}-\omega_{\mathbf{p}}t)},$$

$$\mathbf{E}_{\mathbf{k}_-} = \mathbf{E}^*_{\mathbf{k}}, \qquad \mathbf{k}_- = -\mathbf{k}, \qquad \omega_{\mathbf{k}_-} = -\omega_{\mathbf{k}}, \qquad \omega_{\mathbf{k}} > 0, \tag{1}$$

where $\omega_{\mathbf{p}}$ is the real part of the frequency; $\mathbf{E}_{\mathbf{p}}(t)$ is the amplitude, whose dependence on time is determined, first, by the increment

$$\gamma_{\mathbf{k}} = \frac{\pi}{2} k \left(\frac{\omega_{\mathbf{k}}}{k} \right)^3 \left. \frac{df}{dv} \right|_{v=\omega/k}, \tag{2}$$

where, in place of the unperturbed function, there stands the slowly varying mean distribution function (1–3), and, second, by the nonlinear interaction between harmonics. The "dynamical" equation for $\mathbf{E}_{\mathbf{p}}(t)$ has the form (4,5):

$$\frac{d\mathbf{E}_{\mathbf{p}}}{dt} = \gamma_{\mathbf{p}} \mathbf{E}_{\mathbf{p}} + \sum_{\mathbf{p}'+\mathbf{p}''=\mathbf{p}} V_{\mathbf{p}\mathbf{p}'\mathbf{p}''} \mathbf{E}_{\mathbf{p}'} \mathbf{E}_{\mathbf{p}''} \exp\left[-i(\omega_{\mathbf{p}'} + \omega_{\mathbf{p}''} - \omega_{\mathbf{p}})t\right] +$$

$$+ \sum_{\mathbf{p}'+\mathbf{p}''+\mathbf{p}'''=\mathbf{p}} V_{\mathbf{p}\mathbf{p}'\mathbf{p}''\mathbf{p}'''} \mathbf{E}_{\mathbf{p}'} \mathbf{E}_{\mathbf{p}''} \mathbf{E}_{\mathbf{p}'''} \exp\left[-i(\omega_{\mathbf{p}'} + \omega_{\mathbf{p}''} + \omega_{\mathbf{p}'''} - \omega_{\mathbf{p}})t\right], \tag{3}$$

where the matrix elements are expressed through the distribution function (see below). In a turbulent plasma the phases of the amplitudes $\mathbf{E}_{\mathbf{p}}(t)$ change much faster than their moduli and are correlated with one another during a very short time in comparison with the characteristic time of change of $N_{\mathbf{p}}(t) = |\mathbf{E}_{\mathbf{p}}(t)|^2$. Therefore, in calculating $dN_{\mathbf{p}}/dt$ one may average over the phases of all $\mathbf{E}_{\mathbf{p}}$ at a given instant. Carrying out this averaging in the same way as in (4), we obtain a kinetic— wave equation for waves

$$\frac{dN_p}{dt} = 2\gamma_{pN}p + N_p \left\{ 8 \sum_{p'+\mathbf{p}''=\mathbf{p}} N_{p'} \mathcal{P} \frac{\operatorname{Im}\left(V_{pp'p''} V_{p''p'-p}\right)}{\omega_{p'} + \omega_{p''} - \omega_p} + \right.$$

$$\left. +6 \operatorname{Re} \left(\sum_{p'} N_{p'} V_{pp-p'p'-} \right) \right\} + 4\pi \sum_{p'+\mathbf{p}''=\mathbf{p}} \left\{ N_{p'} N_{p''} \left| V_{pp'p''} \right|^2 + \right.$$

$$\left. +2N_{pN_{p'}} \operatorname{Re}\left(V_{pp'p''} V_{p''p'-p}\right) \right\} \delta(\omega_{p'} + \omega_{p''} - \omega_p), \tag{4}$$

where $\mathcal{P}$ is the principal-value symbol. (4) differs from the corresponding equation (2.16) of work (4) by the presence of the first two terms, which describe the emission and absorption of waves "by the background," in which the mean distribution function of the plasma changes. The third term describes the nonlinear interaction between harmonics that is not connected with their absorption or

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* Another approach to the solution of this circle of questions is being developed in (6).

emission. In a transparent medium γ_p, $\operatorname{Im} V_{pp'p''}$, $\operatorname{Re} V_{pp-p'p'}$ vanish, and (4) goes over into the kinetic equation obtained in (^{4}).

Let us now consider the plasma distribution function $F(\mathbf{r}, \mathbf{v}, t)$. We shall seek it in the form:

$$F(\mathbf{r}, \mathbf{v}, t) = f(\mathbf{v}, t) + \sum_p \exp[i\mathbf{pr}]\, F_p(v, t), \tag{5}$$

$$F_p(v,t) = \frac{e}{im} E_{pf} p(v,t) \exp[-i\omega_{pt}] +$$

$$+ \sum_{p'+\mathbf{p}''=\mathbf{p}} \left(\frac{e}{im}\right)^2 E_{p'} E_{p''} f_{pp'p''} \exp[-i(\omega_{p'} + \omega_{p''})t] +$$

$$+ \sum_{p'+\mathbf{p}''+\mathbf{p}'''=\mathbf{p}} \left(\frac{e}{im}\right)^3 E_{p'} E_{p''} E_{p'''} f_{pp'p''p'''} \exp[-i(\omega_{p'} + \omega_{p''} + \omega_{p'''})t], \tag{6}$$

where $f, f_p, f_{pp'p''}$, etc. are slowly varying functions of time. Substituting (5) into the kinetic equation for the distribution function F and averaging over the phases of the field amplitudes E_p, we obtain

$$\frac{\partial f}{\partial t} = -\frac{e}{m} \sum_p E_p^* \frac{\overline{dF_p}}{\partial \mathbf{v}} \exp[i\omega_{pt}], \tag{7}$$

$$\frac{\partial F_p}{\partial t} + i(\mathbf{pv})F_p = -\frac{e}{m} \frac{\partial f}{\partial \mathbf{v}} E_p \exp[-i\omega_{pt}] -$$

$$-\frac{e}{m} \sum_{p'+\mathbf{p}''=\mathbf{p}} E_{p'} \frac{\partial F_{p''}}{\partial \mathbf{v}} \exp[-i(\omega_{p'} + \omega_{p''})t]. \tag{8}$$

Substituting (6) into (8), we shall have

$$f_p = (\omega_p - \mathbf{pv} + i\gamma)^{-1} \frac{\mathbf{p}}{p} \frac{\partial f}{\partial \mathbf{v}},$$

$$f_{pp'p''} = (\omega_{p'} + \omega_{p''} - \mathbf{pv} + i\varepsilon)^{-1} \left\{ \frac{1}{2} \left[\frac{\mathbf{p}'}{p'} \frac{\partial}{\partial \mathbf{v}} \left(\frac{1}{\omega_{p''} - \mathbf{p}''\mathbf{v} + i\gamma} \frac{\mathbf{p}''}{p''} \frac{\partial f}{\partial \mathbf{v}} \right) + \right.\right.$$

$$\left.\left. + \frac{\mathbf{p}''}{p''} \frac{\partial}{\partial \mathbf{v}} \left(\frac{1}{\omega_{p'} - \mathbf{p}'\mathbf{v} + i\gamma} \frac{\mathbf{p}'}{p'} \frac{\partial f}{\partial \mathbf{v}} \right) \right] + \frac{m}{e} \frac{V_{pp'p''}}{\omega_p - \mathbf{pv} + i\gamma} \frac{\mathbf{p}}{p} \frac{\partial f}{\partial \mathbf{v}} \right\} \tag{9}$$

and so on.

According to (^{5}), the matrix elements $V_{pp'p''}$, $V_{pp'p''p'''}$, which appear in the kinetic equation for waves, are expressed in terms of the corresponding functions in (6):

$$V_{pp'p''} = \sum_{j=e,i} \frac{e_j}{m_j} \omega_{0j}^2 \int \frac{\mathbf{pv}}{p} f^{\,j}_{pp'p''}(\mathbf{v})\, d\mathbf{v}; \tag{10}$$

$$V_{pp'p''p'''} = -i \sum_{j=e,i} \left(\frac{e_j}{m_j}\right)^2 \omega_{0j}^2 \int \frac{\mathbf{pv}}{p} f^{\,j}_{pp'p''p'''}(\mathbf{v})\, d\mathbf{v}, \tag{10a}$$

where ω_{0j} is the plasma frequency; the summation is over particle species.

Let us now substitute (6) into (7) and retain the terms quadratic in $N_p = E_p^* E_p$. In doing so one should bear in mind that the term $\overline{E_p E_{p'} E_{p''}}$ may be regarded as equal to zero only up to terms of third order in E_p. In the next order it is equal to

$$\overline{E_p E_{p'} E_{p''}} = 2\, \frac{\exp[i(\omega_{p'} + \omega_{p''} - \omega_p)t]}{i(\omega_{p'} + \omega_{p''} - \omega_p + i\varepsilon)} \Big(V_{p-p'p''} N_{p'} N_{p''} +$$

$$+ V_{p'p''-p} N_{pN_{p''}} + V_{p''pp'-} N_{pN_{p'}} \Big). \tag{11}$$

To obtain (11) one may proceed as follows: differentiating $E_{pE_{p'}} E_{p''}$ with respect to time, substitute the derivatives from the dynamical equation (3), then average over phases and integrate from $-\infty$ to t.

As a result, the following equation for $f(v,t)$ is obtained:

$$\frac{\partial f}{\partial t} = \left(\frac{e}{m}\right)^2 \sum_p N_p \frac{\mathbf{p}}{p} \frac{\partial}{\partial v} \left[\frac{\gamma_p}{(\omega_p - \mathbf{p}v)^2 + \gamma_p^2} \frac{\mathbf{p}}{p} \frac{\partial f}{\partial v} \right] +$$

$$+ 2 \left(\frac{e}{m}\right)^3 \operatorname{Im} \sum_{p'+p''=p} \frac{\mathbf{p}}{p} \frac{\partial f_{pp'p''}}{\partial v} \frac{V^*_{pp'p''} N_{p'} N_{p''} + 2V_{p''p-p'} N_{pN_{p'}}}{\omega_{p'} + \omega_{p''} - \omega_p + i\varepsilon} +$$

$$+ 3 \left(\frac{e}{m}\right)^4 \operatorname{Im} \sum_{p,p'} \frac{\mathbf{p}}{p} \frac{\partial f_{pp-p'p'}}{\partial v} N_{pN_{p'}}. \tag{12}$$

(12) and (4), together with (9), constitute the complete system of equations of quasilinear theory, supplemented by allowance for the interaction of waves of second order. The terms in (12) and (4) that are quadratic in N_p become important when the increment (2) becomes sufficiently small owing to

> quasilinear relaxation (formation of "plateaus" (1–3)). If the frequency spectrum of the waves is such that the "decay" conditions are satisfied, i.e. $\omega(\mathbf{k}_1) + \omega(\mathbf{k} - \mathbf{k}_1) = \omega(\mathbf{k})$, then the terms standing at the δ-function in (4) lead to a transfer of energy into the region of small phase velocities, where wave absorption is large. If, however, the decay conditions cannot be satisfied (for example, for electron Langmuir oscillations), then the second term in (4) and the corresponding one in (12) play an essential role. It will be shown below that it also can lead to absorption of waves by particles with small velocities $v \ll v_T$ and to the corresponding heating of the plasma; moreover, this effect is much more substantial than those effects of wave interaction which were estimated in (^{3}).

It may be verified that, as in ordinary quasilinear theory, the system (4), (12) possesses an energy integral, i.e.

$$\frac{dW}{dt} = \frac{d}{dt}(W_k + W_f) = 0, \qquad W_k = \sum_{e,i} \frac{nm}{2} \int v^2 f(v)\, d^3v,$$

$$W_f = \frac{1}{8\pi} \sum_p |E_p|^2. \tag{13}$$

Let us now consider, as an example, the relaxation of electron oscillations in a sufficiently rarefied plasma, where collisions may be completely neglected; moreover, we shall regard the width of the wave packet as sufficiently small ($\Delta k \ll k$), and the wave energy as large in comparison with the mean kinetic energy of the particles. Then two stages of the oscillation relaxation process can be indicated. First a "plateau" is formed in the resonant range of velocities $v \sim \omega/k \gg v_T$ (v_T is the thermal velocity of the electrons). This stage of the process is described by ordinary quasilinear theory (1–3). By the time

formation of the energy plateau of the waves will still be large compared with the thermal noise. If the role of Coulomb collisions in the absorption of waves may be neglected, then the relaxation will be determined by the second term in (4) (the third vanishes because of the δ-function).

The imaginary part of the matrix elements, calculated with the aid of (10) and (9), is determined by half-residues of two types. These are, first, terms proportional to df/dv at $v = \omega/k$ and vanishing after the formation of the plateau. Second, some of the matrix elements ($V_{kk'k''}, V_{k''k'k}$) contain imaginary terms proportional to f'_v at

$$v = \frac{\omega_{k'} - \omega_{k''}}{k' - k''} \sim v_T \frac{v_T k}{\omega_0} \ll v_T,$$

thanks to which the damping of the oscillations does not disappear even after the cessation of quasilinear relaxation. Terms of this type have a simple physical meaning. As a result of the nonlinear interaction, the natural plasma oscillations (1) give rise to forced oscillations with combination frequencies $\omega_{k'} \pm \omega_{k''}$ and wave vectors $k' \pm k''$, which effectively interact with particles having velocities

$$v = \frac{\omega_{k'} \pm \omega_{k''}}{|k'| \pm |k''|}.$$

This interaction leads to additional absorption.

Substitution of (10), (9) into (4) leads to the equation

$$\frac{dN_k}{dt} = N_k \sum_{k'} \alpha_{kk'} N_{k'}, \tag{14}$$

where $\alpha_{kk'}$, for $\Delta k = k - k' \ll k$, depends only weakly on k' and is approximately equal to

$$\alpha_{kk'} \sim \left(\frac{e}{m}\right)^2 \frac{v_T^2 k^2}{\omega_0^3} \left(\frac{\Delta k}{k}\right)^2 f'_{v=u}, \qquad u \simeq v_T \frac{v_T k}{\omega_0}, \qquad f'_{v=u} \simeq -\frac{k}{v_T \omega_0}. \tag{15}$$

Owing to this one may write

$$\frac{dW_f}{dt} \simeq -BW_f^2, \qquad B \sim \frac{v_T k_0^3}{mn\omega_0^2} \left(\frac{\Delta k}{k_0}\right)^2, \tag{16}$$

where W_f is the total energy density of the turbulent pulsations, and k_0 is the mean wave number. For $W_f/nT \ll 1$, the quantity B may be considered independent of time. Then the solution of (16) will have the form

$$W_f(t) \simeq \frac{W_{0f}}{1 + BW_{0f}t}, \tag{17}$$

where W_{0f} is the energy of the oscillations at the moment of plateau formation, equal in order of magnitude to the initial energy of the oscillations. Thus the characteristic relaxation time is equal to

$$\tau \sim (BW_{0f})^{-1} \sim \left(\frac{v_f}{v_T}\right)^3 \left(\frac{k_0}{\Delta k}\right)^2 \frac{nT}{W_{0f}} \omega_0^{-1}, \qquad v_f = \frac{\omega_0}{k}.$$

In conclusion I express my deep gratitude to R. Z. Sagdeev for numerous fruitful discussions and to A. A. Galeev for useful discussion of the results.

Received
28 III 1963

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