MATHEMATICS

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Abstract

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AN ALGORITHM FOR OPTIMIZING THE PRODUCTION AND DISTRIBUTION OF HETEROGENEOUS PRODUCTS

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We consider the problem of the optimal assortment loading of enterprises, taking into account the following factors:

a) the cost of a unit of resource r at production points i ($c_{ir}^{(1)}$);

b) the consumption of resource r at production point i per unit of product p (γ_{ipr});

c) upper bounds on the total use of resource r at production point i (b_{ir});

d) the demand of consumer k (market conditions) for product p (d_{kp});

e) the cost of delivering a unit of product p by transport t on the route i—k ($c_{iktp}^{(2)}$);

f) the productivity of transport t on the route i—k, in units of product p (δ_{iktp});

g) the total number of units of transport t (a_t);

h) x_{iktp} is the quantity of product p transported on the route i—k by transport t.

Mathematical model:

Find

$$\min \sum_{i,k,t,p} c_{iktp} x_{iktp} \tag{1}$$

under the conditions $x_{iktp} \geqslant 0$ and

$$\sum_{k,t,p} x_{iktp} \gamma_{ipr} \leqslant b_{ir}; \tag{2}$$

$$\sum_{i,k,p} \frac{x_{iktp}}{\delta_{iktp}} \leqslant a_t, \qquad \text{all } \delta_{iktp} \geqslant 0; \tag{3}$$

$$\sum_{i,t} x_{iktp} = d_{kp}, \tag{4}$$

where $i = 1, \dots, i_*$; $k = 1, \dots, k_*$; $t = 1, \dots, t_*$; $p = 1, \dots, p_*$;

$$r = 1, \dots, r_*; \qquad c_{iktp} = \sum_{r=1}^{r_*} c_{ir}^{(1)} \gamma_{ipr} + c_{iktp}^{(2)}.$$

If all costs involved in locating production are taken to be linear, then the model described can be regarded as a location model.

Consider constraints (3) and (4). They form a convex bounded polyhedron B with a finite number of vertices h. Denote them by $\bar{\xi}^{(\sigma)}$, where $\sigma = 1, \dots, h$. Then, if $\bar{x} \in B$, the representation

$$\bar{x} = \sum_{\sigma=1}^{h} \bar{\xi}^{(\sigma)} \lambda_\sigma, \tag{5}$$

where

$$\lambda_\sigma \geqslant 0, \qquad \sum_{\sigma=1}^{h} \lambda_\sigma = 1 \tag{6}$$

or

$$x_{iktp} = \sum_{\sigma=1}^{h} \xi_{iktp}^{(\sigma)} \lambda_\sigma. \tag{5'}$$

Substituting $(5')$ into (1) and (2), we obtain the problem:

Find the minimum of the linear form

$$\sum_{i,k,t,p,\sigma} c_{iktp} \xi_{iktp}^{(\sigma)} \lambda_\sigma \tag{7}$$

subject to the conditions

$$\sum_{k,t,p,\sigma} \xi_{iktp}^{(\sigma)} \lambda_\sigma \gamma_{ipr} \leqslant b_{ir}. \tag{8}$$

From the duality theorem of linear programming and the modified simplex method there follow the optimality conditions for problem (7) and (8): in order that the basic vector $(\lambda_{\sigma_1}, \ldots, \lambda_{\sigma_{i_*+1}})$ deliver the minimum of (7) under constraints (8), it is necessary and sufficient that there exist an evaluation vector $\bar{\pi}^{(1)} = (\pi_0^{(1)}, \pi_{11}^{(1)}, \ldots, \pi_{i_* r_*}^{(1)})$ such that the conditions

$$\sum_{i,k,t,p} \left[\sum_{r=1}^{r_*} \left(c_{ir}^{(1)} - \pi_{ir}^{(1)} \right) \gamma_{ipr} + c_{iktp}^{(2)} \right] \xi_{iktp}^{(\sigma)} \geqslant \pi_0^{(1)} \tag{9}$$

hold for all $\sigma = 1, \ldots, h$.

For conditions (9) to be fulfilled, it is necessary and sufficient that the relations

$$\min \sum_{i,k,t,p} \left[\sum_{r=1}^{r_*} \left(c_{ir}^{(1)} - \pi_{ir}^{(1)} \right) \gamma_{ipr} + c_{iktp}^{(2)} \right] x_{iktp} \geqslant \pi_0^{(1)}, \tag{10}$$

where all $x_{iktp} \geqslant 0$ are subject to constraints (3), (4), be fulfilled.

We shall call problem (10), (3), (4) criterion problem I.

Let us first consider several possible simplifications.

1) At each production point i, the resource of any kind r, except for one r_i, is available in such a quantity that the constraints on the use of resource $r \neq r_i$ become inessential. It is not difficult to see that, for this, it is sufficient that the conditions

$$\sum_{k,l,p} \min \left(a_t \delta_{iktp},\, d_{kp} \right) \gamma_{ipr} \leqslant b_{ir}, \qquad r \neq r_i \tag{11}$$

be fulfilled. Then conditions (2) are written in the form

$$\sum_{k,t,p} x_{iktp} \gamma_{ipr_i} \leqslant b_{ir_i}. \tag{2'}$$

Constraints of this kind were encountered by the author in solving the problem of locating the paper industry in the long term up to 1980. The conditions (10) of the criterion problem take the form

$$\min \sum_{i,k,t,p} \left(c_{iktp} - \pi_{ir_i}^{(1)} \gamma_{ipr_i} \right) x_{iktp} \geqslant \pi_0^{(1)}. \tag{10'}$$

2) If all types of products are such that one can choose for them a common unit of measurement from the standpoint of transportation (volume, weight), and the productivity of transport t does not depend on the route, i.e., all

$\delta_{iktp} = \delta_t$, then conditions (3) are written as follows:

$$\sum_{i,k,p} x_{iktp} \leqslant a_t \delta_t. \tag{3'}$$

The proposals made above do not, however, preclude taking into account the efficiency of using transport t on routes i—k for transporting product p, since information on the efficiency of each variant of use is incorporated in c_{iktp}.

Simplifications 1) are in principle inessential from the standpoint of the algorithm considered for solving the problem, but they considerably facilitate the computation, since in this case the total number of elementary operations is reduced by several orders if the number of resources and production points is large. Simplifications 2) make it possible to reduce the solution of the criterion problem (10), (3′), (4) to the solution of a single transportation problem. Indeed, let

$$q_{ktp} = \min_{1 \leqslant i \leqslant i_*} \left(c_{iktp} - \sum_{r=1}^{r_*} \pi_{ir}^{(1)} \gamma_{ipr} \right) = c_{i_1 ktp} - \sum_{r=1}^{r_*} \pi_{i_1 r}^{(1)} \gamma_{i_1 pr}.$$

Then the criterion problem (10), (3′), (4) reduces to the following:

Find

$$\min \sum_{k,t,p} q_{ktp} y_{ktp} \tag{12}$$

subject to

$$y_{ktp} \geqslant 0;$$

$$\sum_{k,p} y_{ktp} \leqslant a_t \delta_t; \tag{13}$$

$$\sum_{t} y_{ktp} = d_{kp}, \tag{14}$$

where

$$x_{iktp} = \begin{cases} y_{ktp}, & \text{if } i = i_1, \\ 0, & \text{if } i \neq i_1. \end{cases} \tag{15}$$

Problem (12)–(14), by the simple replacement of the indices (k, p) by a single index

$$s = k + k_*(p-1),$$

reduces to an open transportation problem.

In the case where simplifications 2) are impossible, the solution of the criterion problem is carried out by the Dantzig–Wolfe decomposition method. Indeed, considering conditions (4) as defining a polyhedron G with extreme points $\bar{\eta}^{(\nu)}$, we obtain the following:

if $\bar{x} \in G$, then the representation

$$\bar{x} = \sum_{\nu=1}^{g} \bar{\eta}^{(\nu)} \mu_\nu \tag{16}$$

is possible, or

$$x_{iktp} = \sum_{\nu=1}^{g} \eta_{iktp}^{(\nu)} \mu_\nu, \tag{16'}$$

where

$$\sum_{\nu=1}^{g} \mu_\nu = 1, \qquad \mu_\nu \geqslant 0. \tag{17}$$

Substituting (16′) into the left-hand side of inequality (10) and into (3), we obtain the following problem:

Find the minimum

$$\sum_{\nu=1}^{g} \alpha^{(\nu)} \mu_\nu \tag{18}$$

subject to

$$\sum_{\nu=1}^{g} S_t^{(\nu)} \mu_\nu \le a_t, \qquad \sum_{\nu=1}^{g} \mu_\nu = 1, \qquad \mu_\nu \ge 0, \tag{19}$$

where

$$\alpha^{(\nu)} \equiv \sum_{i,k,t,p} \left(c_{iktp} - \sum_{r=1}^{r_*} \pi_{ir}^{(1)} \gamma_{ipr} \right) \eta_{iktp}^{(\nu)}; \tag{20}$$

$$S_t^{(\nu)} \equiv \sum_{i,k,p} \frac{x_{iktp}}{\delta_{iktp}}. \tag{21}$$

From the duality theorem of linear programming, the modified simplex method, and the ideas of block programming it follows that, in order for the basic vector $(\mu_{\nu_1}, \dots, \mu_{\nu_{t_*+1}})$ to deliver a minimum to functional (18) under constraints (19), it is necessary and sufficient that there exist a price vector $\bar{\pi}^{(2)} = (\pi_0^{(2)}, \pi_1^{(2)}, \dots, \pi_{t_*}^{(2)})$ for which the following relations hold:

$$\min \sum_{i,k,t,p} \left(c_{iktp} - \sum_{r=1}^{r_*} \pi_{ir}^{(1)} \gamma_{ipr} - \frac{\pi_t^{(2)}}{\delta_{iktp}} \right) x_{iktp} \geq \pi_0^{(2)} \tag{22}$$

under the following constraints on nonnegative x_{iktp}:

$$\sum_{i,t} x_{iktp} = d_{kp}. \tag{23}$$

The solution of criterion problem II with conditions (22), (23) is easily written in analytic form:

$$\min \sum_{i,k,t,p} \left(c_{iktp} - \sum_{r} \pi_{ir}^{(1)} \gamma_{ipr} - \frac{\pi_t^{(2)}}{\delta_{iktp}} \right) x_{iktp} = \sum_{k,p} e_{kp} d_{kp},$$

where

$$e_{kp} \equiv \min_{\substack{1 \leq i \leq i_* \\ 1 \leq t \leq t_*}} \left(c_{iktp} - \sum_{r} \pi_{ir}^{(1)} \gamma_{ipr} - \frac{\pi_t^{(2)}}{\delta_{iktp}} \right) = c_{i_1 k t_1 p} - \sum_{r} \pi_{i_1 r}^{(1)} \gamma_{i_1 pr} - \frac{\pi_{t_1}^{(2)}}{\delta_{i_1 k t_1 p}},$$

and

$$x_{iktp} = \begin{cases} d_{kp}, & \text{if } i = i_1,\ t = t_1, \\ 0 & \text{in all other cases.} \end{cases}$$

If condition (10) is satisfied, then the optimal vector $(\lambda_{\sigma_1}, \dots, \lambda_{\sigma_{t_*+1}})$ has been found, and from (5′) the solution of problem (1)–(4) is easily obtained. If, however, (10) is not satisfied, then on the basis of the obtained solution of criterion problem I the column to be introduced into the basis is computed, along with a new basic solution $(\lambda_{\sigma_1'}, \dots, \lambda_{\sigma_{t_*+1}'})$, and a new price vector $\bar{\pi}^{(1)'}$. All initial values—the basic columns, the basic solution, the vector of basic prices, and the price vector—are obtained from the general ideas of linear programming by introducing artificial variables.

Everything said concerning the initial data of criterion problem I also applies to criterion problem II.

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Note: Figure translations are in progress. See original paper for figures.