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Abstract

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ON CHEBYSHEV AND ALMOST-CHEBYSHEV SUBSPACES

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1. A subspace l of a Banach space B is called **Chebyshev** if for every element $x \in B$ there exists in the subspace l a unique nearest element (element of best approximation) ([1]).

As is known, there exist Banach spaces possessing no Chebyshev subspace of finite dimension or finite index ([2]). Moreover, the following turns out to be true.

Theorem 1. *There exists a (nonseparable) Banach space possessing no Chebyshev subspace at all.*

Chebyshev subspaces of finite dimension may fail to exist, as is known, also in separable spaces (including the classical ones ([2,3])). At the same time we have

Theorem 2. *There exists an infinite-dimensional separable Banach space that has no Chebyshev subspace of infinite dimension.*

There is a certain connection between the question of the existence of Chebyshev subspaces and the presence of extreme points of the unit sphere S_B of the space B. Thus, *for the existence of a Chebyshev subspace having a reflexive complement, it is necessary that there be extreme points of the sphere S_B.* (The condition of reflexivity of the complement in this assertion cannot be omitted.) In particular, *Chebyshev subspaces of index $n < \infty$ can exist only in a space whose unit sphere has at least n extreme points.*

2. In connection with the facts cited, we introduce the following definition. A subspace l of a Banach space B is called **almost Chebyshev** if it possesses a unique nearest element for all $x \in B$, with the possible exception of elements forming in the space B a set only of the first category. It is not difficult to give an interpretation of this concept (as well as of the subsequent theorems) in terms of the geometry of the sphere of the space B.

We next investigate the question of the existence of almost-Chebyshev subspaces.

Denote by $L(B, l_0)$ the metric space formed by the set of all subspaces $l \subset B$ that are isomorphic (in the Banach sense) to the subspace $l_0 \subset B$, with metric

$\rho(l_1, l_2)$ defined by the relation

$$\rho(l_1, l_2) = \inf_T \left\{ \sup_{\substack{x \in l_1 \\ x \neq 0}} \left\| \frac{x}{\|x\|} - \frac{Tx}{\|Tx\|} \right\| + \ln \max \left[\|T\|, \|T^{-1}\| \right] \right\}, \qquad (*)$$

where the infimum is taken over all isomorphisms T of the subspace l_1 onto l_2.

Lemma 1. *The metric space $L(B, l_0)$ is complete.*

Theorem 3 (on the existence of almost-Chebyshev subspaces). *Whatever reflexive subspace l_0 of the separable Banach space B may be, the collection of almost-Chebyshev subspaces isomorphic to l_0 constitutes in the metric space $L(B, l_0)$ an everywhere dense set of the second category.*

Thus, in any separable Banach space there exist almost Chebyshev subspaces of any finite dimension.

The conditions of separability of the space B and reflexivity of the subspace l_0, as the following theorems show, cannot be omitted.

Theorem 4 (strengthening of Theorem 1). *There exists a nonseparable Banach space possessing no almost Chebyshev subspace.*

Theorem 5. *There exists a separable (nonreflexive) Banach space possessing no nonreflexive almost Chebyshev subspace.*

If the space B is conjugate to some space ${}^*B$, then the condition of reflexivity of l_0 can be weakened.

A linear operator mapping a space $l \subset B$ into the space B will be called **regularly continuous** if it is continuous in the topology $\sigma(B, {}^*B)$. We shall call subspaces l_1 and l_2 **regularly isomorphic** if there exists an isomorphism T, regularly continuous in both directions, of the subspace l_1 onto l_2. By $L^*(B, l_0)$ we shall denote the metric space formed by all subspaces $l \in B$ regularly isomorphic to the regularly closed subspace $l_0 \in B$, with a metric analogous to $(*)$, the only difference being that the inf is taken in this case only over regularly continuous isomorphisms T.

Lemma 1*. *The metric space $L^*(B, l_0)$ is complete.*

Theorem 3*. *Whatever the regularly closed subspace l_0 of the conjugate separable space B, the collection of almost Chebyshev subspaces regularly isomorphic to l_0 constitutes, in the space $L^*(B, l_0)$, an everywhere dense set of the second category.*

In addition to this theorem we also note that in a conjugate separable space there always exist almost Chebyshev subspaces of index $n < \infty$. At the same time a nonconjugate separable space may have no almost Chebyshev subspace of finite index (for example, if the unit sphere of this space has no extreme points).

In the proof of Theorems 3 and 3*, besides Lemmas 1 and 1*, we used some general propositions on isomorphic subspaces. We give, in particular, the following lemma.

Lemma 2. *Subspaces of the same finite index are isomorphic. Moreover, if l_1 and l_2 are subspaces of unit index, given by the equations $f(x) = 0$, $\varphi(x) = 0$ ($f, \varphi \in B^*$, $\|f\| = \|\varphi\| = 1$, $x \in B$), then for every $\varepsilon > 0$ there exists an isomorphism T_ε of the subspace l_1 onto l_2 such that*

$$\max\{\|T_\varepsilon\|,\ \|T_\varepsilon^{-1}\|\} \leqslant 1 + (2+\varepsilon)\|f-\varphi\|.$$

3. Let us consider the question of the characteristic properties of finite-dimensional almost Chebyshev subspaces in the space $C(Q)$ of continuous real functions defined on an arbitrary metric compactum Q (as is known ([4]), if the compactum Q is not homeomorphic to a part of a circle, then the space $C(Q)$ has no Chebyshev subspaces of finite dimension $n > 1$). A system of n linearly independent functions $S_n = \{f_1(q), \dots, f_n(q)\}$ ($q \in Q$) will be called **almost Chebyshev** if the subspace l_n spanned by these functions is almost Chebyshev. Elements of the subspace l_n, as usual, will be called polynomials with respect to the system of functions S_n.

We shall call a subcompactum $\widetilde{Q} \subset Q$ **essential** if it is the closure of an open set of the space Q. To each subcompactum $\widetilde{Q}$ we assign the number $N(\widetilde{Q})$, equal to the number of points of the subcompactum $\widetilde{Q}$ if there are no more than n of them, and equal to n otherwise.

Theorem 5. *In order that the system of functions $S_n = \{f_1(q), \dots, f_n(q)\}$ be almost Chebyshev on the compactum Q, it is necessary and sufficient that*

on every essential subcompact $\widetilde{Q} \subset Q$ no more than $N(\widetilde{Q})$ linearly independent polynomials with respect to the system S_n vanish identically.

The conditions of the theorem are obviously simplified for the case of a perfect compactum; in particular, if Q is a segment, then they reduce to the requirement of strengthened linear independence of the functions $f_1(q), \dots, f_n(q)$ ([3]).

A sequence of functions $f_1(q), \dots, f_n(q), \dots$ will be called almost Markovian (cf. ([5])) if, for every n, the system $f_1(q), \dots, f_n(q)$ is almost Chebyshev. For the space $C(Q)$ one can give a constructive proof of the existence not only of almost Chebyshev systems, but also of almost Markovian systems.

Lemma 3. Whatever the metric compactum Q, one can construct a continuous function $\alpha(q)$ which is not identically constant on any essential subcompactum containing at least two points.

Theorem 6. Let $\alpha(q)$ be the function indicated in Lemma 3; then the sequence of functions

$$1,\ \alpha(q),\ [\alpha(q)]^2,\ \dots,\ [\alpha(q)]^n,\ \dots$$

is almost Markovian.

In some questions, for example in interpolation problems on compacta, almost Chebyshev systems are, to a certain extent, able to compensate for the absence of Chebyshev systems usually used for interpolation on a segment.

Let Q^n be the n-fold product of the compactum Q with itself.

Theorem 7. Systems of n points $q_1, q_2, \dots, q_n$ ($q_k \in Q$) for which the Lagrange interpolation problem with respect to an almost Chebyshev system of functions has a unique solution form, in the space Q^n, an everywhere dense set of the second category.

Let us note in conclusion that P. P. Korovkin' s theorem ([6]) on the completeness of a Markovian system on a certain perfect set $M \subset Q$ carries over to almost Markovian systems.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.