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Abstract

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MATHEMATICS

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ON THE PROPERTIES OF SUMS OF CERTAIN TRIGONOMETRIC SERIES

(Presented by Academician V. I. Smirnov on 31 I 1963)

In the present paper we investigate certain properties of sums of cosine and sine series. Many authors have dealt with this problem. The works of L. Fejér ([1]), who posed and solved a number of problems on finding properties of sums of the indicated series, deserve special attention. However, many questions of this broad problem still remain unsolved.

Lemma 1. *For a sequence $\{c_n\}$ with bounded variation and $c_n \to 0$, there exists a sequence $\{d_n\}$ such that*

$$c_n = d_n + d_{n+1}, \qquad d_n \to 0. \tag{1}$$

The proof follows from Lemma 1 ([5]), if one takes into account that $c_n = p_n - q_n$, where $p_n \downarrow 0$ and $q_n \downarrow 0$ ($d_n = \Delta c_n + \Delta c_{n+2} + \Delta c_{n+4} + ...$).

Lemma 2. *For $\{c_n\}$ with bounded variation and $c_n \to 0$, a necessary and sufficient condition for the k-fold monotonicity of $\{d_n\}$, where $c_n = d_n + d_{n+1}$, is*

$$\Delta^{k+1} c_n + \Delta^{k+1} c_{n+1} + \Delta^{k+1} c_{n+4} + ... \geq 0 \qquad (n = 1, 2, ...). \tag{2}$$

Condition (2) is intermediate between the conditions $\Delta^k c_n \geq 0$ and $\Delta^{k+1} c_n \geq 0$.

In ([1]) Fejér proved that if $\{b_n\}$ is convex (twice monotone) and $b_n \to 0$, then

$$\sum_{n=1}^{\infty} b_n \sin nx > 0, \qquad 0 < x < \pi.$$

This assertion is also valid under somewhat weaker restrictions. It is obtained from the following theorem, in whose proof Lemmas 1 and 2 play an essential role.

Theorem 1. *If $\Delta^2 b_n + \Delta^2 b_{n+2} + \Delta^2 b_{n+4} + ... \geq 0$ $(n = 2, 3, ...)$ and $b_n \to 0$, then the following two relations hold:*

$$\sum_{n=1}^{\infty} b_n \sin nx \geq \beta_1 \sin x, \qquad 0 \leq x \leq \pi; \tag{3}$$

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$$\sum_{n=1}^{\infty} b_n \sin nx \geq \Delta b_1 \sin x + 2\beta_2 \cos\frac{x}{2}\sin\frac{3}{2}x, \qquad 0 \leq x \leq \pi; \tag{4}$$

where $\beta_1 = \Delta^2 b_1 + \Delta^2 b_3 + \Delta^2 b_5 + \ldots$ *and* $\beta_2 = \Delta^2 b_2 + \Delta^2 b_4 + \ldots$.

The theorem is proved on the basis of the transformation

$$\sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=1}^{\infty} (\bar{b}_n + \bar{b}_{n+1}) \sin nx =$$

$$= \bar{b}_1 \sin x + 2\cos\frac{x}{2}\sum_{n=1}^{\infty} \bar{b}_{n+1} \sin\left(n+\frac{1}{2}\right)x$$

and the relations

$$\sum_{n=1}^{\infty} \bar{b}_{n+1} \sin\left(n+\frac{1}{2}\right)x \geqslant -\bar{b}_2 \sin\frac{x}{2},$$

$$\sum_{n=1}^{\infty} \bar{b}_{n+1} \sin\left(n+\frac{1}{2}\right)x \geqslant \Delta\bar{b}_2 \sin\frac{3}{2}x - \bar{b}_3 \sin\frac{x}{2}, \qquad 0 \leqslant x \leqslant 2\pi.$$

Remark 1. Suppose $\Delta^2 b_n + \Delta^2 b_{n+2} + \cdots \geqslant 0$ also for $n = 1$. Then, by Lemma 2, $\Delta b_1 \geqslant \beta_2 \geqslant 0$. Consequently,

$$\sum_{n=1}^{\infty} b_n \sin nx \geqslant \Delta b_1 \sin x + 2\beta_2 \cos\frac{x}{2}\sin\frac{3}{2}x \geqslant 4\beta_2 \sin x \cos^2\left(\frac{x}{2}\right), \qquad 0 \leqslant x \leqslant \pi. \tag{5}$$

From (3) and (5) it is clear that if $\Delta^2 b_n + \Delta^2 b_{n+2} + \cdots \geqslant 0$ $(n = 1, 2, \ldots)$ and $b_n \to 0$, then the sum of the sine series under consideration is nonnegative on $(0, \pi)$. We note that the condition $\Delta^2 b_n + \Delta^2 b_{n+2} + \cdots \geqslant 0$ cannot be replaced by the condition $\Delta b_n \geqslant 0$.

Remark 2. Suppose, in particular, that $\{b_n\}$ $(n = 1, 2, \ldots)$ is convex, $b_n \to 0$, and not all $b_n = 0$ (this is so if $b_1 \neq 0$). In this case it is obvious that β_1 and β_2 cannot both be equal to zero; then the right-hand side in at least one of the relations (3) or (5) is positive on $(0, \pi)$, whence Fejér' s assertion follows. Moreover, in the case under consideration the relations (3) or (4), (5) give a positive lower bound on $(0, \pi)$.

Remark 3. The numbers β_1 and β_2 can be positive also in the case when $\{b_n\}$ is not convex.

Next, as Fejér showed (1), the sum of the series $\sum_{n=1}^{\infty} a_n \cos nx$ decreases monotonically on $(0, \pi)$ if $a_n \to 0$ and $\Delta^4 a_n \geqslant 0$ $(n = 1, 2, \ldots)$. Consequently, under these conditions

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$$\sum_{n=1}^{\infty} a_n \cos nx \geqslant \sum_{n=1}^{\infty} a_n \cos n\pi.$$

The last inequality is valid also under weaker restrictions; namely, Theorems 2 and 3 hold.

Theorem 2. If $\Delta^4 a_n + \Delta^4 a_{n+2} + \cdots \geqslant 0$ $(n = 1, 2, \ldots)$ and $a_n \to 0$, then

$$\sum_{n=1}^{\infty} a_n \cos nx \geqslant \sum_{n=1}^{\infty} a_n \cos n\pi + 2\alpha \cos^2\left(\frac{x}{2}\right),$$

where

$$\alpha = \sum_{k=1}^{\infty} k\Delta^4 a_{2k-1} \geqslant 0.$$

This theorem is proved with the aid of Lemmas 1 and 2, the transformation

$$\sum_{n=1}^{\infty} a_n \cos nx = -\bar{a}_1 + \operatorname{ctg}\frac{x}{2}\sum_{n=1}^{\infty} \Delta\bar{a}_n \sin nx \qquad (a_n = \bar{a}_n + \bar{a}_{n+1})$$

and Theorem 1.

Remark. The condition $\Delta^4 a_n + \Delta^4 a_{n+2} + \cdots \geqslant 0$ cannot be replaced by the condition $\Delta^3 a_n \geqslant 0$.

Theorem 3. If $\Delta^2 c_n + \Delta^2 c_{n+2} + \cdots \geqslant 0$ $(n = 1, 2, \ldots)$ and $c_n \to 0$, then

$$\sum_{n=1}^{\infty} \frac{c_n}{n} \cos nx \geqslant \sum_{n=1}^{\infty} \frac{c_n}{n} \cos n\pi + 2\beta \cos^2\left(\frac{x}{2}\right),$$

where

$$\beta = \Delta^2 c_1 + \Delta^2 c_3 + \Delta^2 c_5 + \cdots.$$

Remark. The sequence $\{a_n\} = \{c_n/n\}$, where $\{c_n\}$ satisfies the conditions of Theorem 3, need not satisfy the conditions of Theorem 2. Conversely, the existence of a sequence satisfying the conditions of Theorem 2 but not satisfying the conditions of Theorem 3 is obvious, since in Theorem 3 $a_n = c_n/n = O(n^{-1})$, whereas in Theorem 2 this is not assumed. Thus, Theorems 2 and 3 are incomparable.

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Let us now consider cosine and sine series with a convex sequence of coefficients tending to zero. For them Theorems 4-7 hold; in the proofs of these theorems one mainly uses the Young-Kolmogorov theorem ([2,3]) stating that

$$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx \geqslant 0, \tag{6}$$

if $\{a_n\}$ $(n=0,1,2,\ldots)$ is convex and $a_n \to 0$, and Theorem 1.

Theorem 4. If $\{a_n\}$ $(n=1,2,\ldots)$ is convex and $a_n \to 0$, then

$$\sum_{n=1}^{\infty} a_n \cos nx \leqslant a_1 \cos x+\cdots$$

$$\cdots+a_{N-1} \cos (N-1) x+\frac{a_N}{2} \cos N x, \qquad \frac{\pi}{2N} \leqslant x \leqslant \frac{\pi}{N}.$$

In particular, putting here $N=1$, we obtain

$$\sum_{n=1}^{\infty} a_n \cos nx \leqslant \frac{a_1}{2} \cos x, \qquad \frac{\pi}{2} \leqslant x \leqslant \pi.$$

Theorem 5. If $\{a_n\}$ $(n=1,2,\ldots)$ is convex and $a_n \to 0$, then

$$\sum_{n=1}^{\infty} a_n \cos nx \geqslant a_1 \cos x+\cdots+a_{N-1} \cos (N-1) x+\frac{a_N}{2} \cos N x,$$

$$\frac{3\pi}{2N} \leqslant x \leqslant \frac{2\pi}{N}, \qquad N \geqslant 2. \tag{7}$$

The segments $[3\pi/2N,\, 2\pi/N]$ cover the whole interval $(0,\pi]$ with the exception of the interval $(2\pi/3,\, 3\pi/4)$. A lower bound in the wider interval $\pi/2 \leqslant x \leqslant 3\pi/4$ is given by the following formula:

$$\sum_{n=1}^{\infty} a_n \cos nx \geqslant a_1 \cos x+\frac{a_2}{2} \cos^2 x-\left(a_2+\frac{a_3}{2} \cos x\right) \sin^2 x.$$

Theorem 6. If $\{b_n\}$ $(n=1,2,\ldots)$ is convex and $b_n \to 0$, then

$$\sum_{n=1}^{\infty} b_n \sin nx \leqslant b_1 \sin x+\cdots+b_{N-1} \sin (N-1) x+\frac{b_N}{2} \sin N x,$$

$$\frac{\pi}{N} \leqslant x \leqslant \frac{3\pi}{2N}, \qquad N \geqslant 2,$$

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$$\sum_{n=1}^{\infty} b_n \sin nx \leqslant b_1 \sin x + \frac{b_2}{2} \sin 2x, \qquad \frac{\pi}{2} \leqslant x \leqslant \pi.$$

Theorem 7. If $\{b_n\}$ $(n = 1, 2, ...)$ is convex and $b_n \to 0$, then

$$\sum_{n=1}^{\infty} b_n \sin nx \geqslant \frac{b_1}{2} \sin x, \qquad 0 < x \leqslant \frac{\pi}{2},$$

$$\sum_{n=1}^{\infty} b_n \sin nx \geqslant \left(\frac{b_1}{2} + b_2 \cos x + \frac{b_3}{2} \cos^2 x\right) \sin x, \qquad \frac{\pi}{2} \leqslant x \leqslant \frac{2}{3}\pi,$$

$$\sum_{n=1}^{\infty} b_n \sin nx \leqslant b_1 \sin x + b_2 \sin 2x + \frac{b_3}{2} \sin 3x, \qquad \frac{2}{3}\pi \leqslant x \leqslant \frac{5}{6}\pi, \tag{8}$$

$$\sum_{n=1}^{\infty} b_n \sin nx \geqslant b_1 \sin x + b_2 \sin 2x + \frac{b_3}{2} \cos x \sin 2x, \qquad \frac{3}{4}\pi \leqslant x \leqslant \pi.$$

Remark 1. By the Young–Kolmogorov theorem (see (6))

$$\sum_{n=1}^{\infty} a_n \cos nx \geqslant -\frac{a_0}{2},$$

if $\{a_n\}$ $(n = 0, 1, 2, ...)$ is convex and $a_n \to 0$. The least value of $a_0/2$ that can be found from the Young–Kolmogorov theorem is equal to $a_0/2 = a_1 - a_2/2$ $(\Delta^2 a_0 = 0)$ and, as we see, does not depend on x. In Theorem 5 the lower bound of the cosine series depends on x.

For example, putting $N = 2$ in (7), we obtain

$$\sum_{n=1}^{\infty} a_n \cos nx \geqslant a_1 \cos x + \frac{a_2}{2} \cos 2x, \qquad \frac{3}{4}\pi \leqslant x \leqslant \pi;$$

$$a_1 \cos x + \frac{a_2}{2} \cos 2x > -a_1 + \frac{a_2}{2} \quad \text{for} \quad \frac{3}{4}\pi \leqslant x < \pi,$$

$$a_1 \cos \pi + \frac{a_2}{2} \cos 2\pi = -a_1 + \frac{a_2}{2}.$$

Remark 2. The right-hand sides in (8) are all positive in the corresponding intervals.

Theorem 8. If $a_n \geqslant 0$, $\{a_n\}$ is of bounded variation and $a_n \to 0$, then the function $C(x)$, where

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$$C(x) = \sum_{n=1}^{\infty} a_n \cos nx,$$

has a root in $(0, \pi)$.

In the proof of this theorem the A-integrability of $C(x)$ is used ((4), p. 659). In the particular case when the series for $C(x)$ is a Fourier–Lebesgue series, the theorem follows, for example, from the equality

$$\int_0^{\pi} C(x)\, dx = 0.$$

Theorem 9. For any segment $[x, \pi]$, $0 < x < \pi$, there exists an infinite set of cosine series with convex coefficient sequences tending to zero whose sums $C(x)$ are negative on $[x, \pi]$.

This theorem is proved on the basis of the fact that $C(x) < 0$ for x satisfying the inequality

$$a_2 \cos^2\left(\frac{x}{2}\right) - a_1 \cos x > 0 \qquad (0 < x \leqslant \pi).$$

Corollary of Theorems 8 and 9. There exist cosine series

$$\sum_{n=1}^{\infty} a_n \cos nx$$

with a convex coefficient sequence tending to zero, whose sums have a root arbitrarily close to zero.

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Note: Figure translations are in progress. See original paper for figures.

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