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Abstract

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ON THE ORDER OF GROWTH OF THE ε-ENTROPY ON SOME COMPACT CLASSES OF FUNCTIONS

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1. In the present note an estimate is given for the growth of the ε-entropy of certain functional compacta defined by their difference properties. The case of compacta defined by first differences of derivatives was studied by A. N. Kolmogorov ([3,5]). The case of differences of second order was studied in ([2,6]). In the present note, classes specified by difference properties of arbitrary order are studied systematically.

Let us agree on notation. Let R^n be n-dimensional Euclidean space. The letters $x, y, z, \ldots$ will denote vectors from R^n; the letters $k, l, m, \ldots$—vectors with integral nonnegative components; $x_i, y_i, z_i, \ldots$ denote the i-th coordinates of the corresponding vectors. Indices referring to vectors will be placed as superscripts (for example, x^m). By e^i we denote the vectors with components δ_j^i, $j = 1, 2, \ldots, n$ (δ_j^i is the Kronecker symbol), $e = \sum_{i=1}^n e^i$.

Let $\omega_{k_i}^{(i)}(f;t)$ be the k_i-th modulus of smoothness of the function $f(x)$, defined in the domain G, taken with respect to the i-th variable, i.e.

$$\omega_{k_i}^{(i)}(f;t) = \sup_{x\in G}\left|\Delta_{i,h}^{k_i} f(x)\right| = \sup_{\substack{x\in G\\ |h|\le t}}\left|\sum_{\nu=0}^{k_i}(-1)^{k_i-\nu}\binom{\nu}{k_i} f(x+\nu h e^i)\right|.$$

We denote by

$$W = \left(\prod_{i=1}^{n}{}^{\otimes}\Delta_i^{k_i}(\varphi_i), \Pi, M\right)^*$$

the class of functions in the n-dimensional parallelepiped $\Pi = \prod_{i=1}^n [a_i, b_i]$ for which the following inequalities hold:

$$\begin{aligned} &1)\quad |f(x)| \le M; \\ &2)\quad \omega_{k_i}^{(i)}(f;t) \le \varphi_i(t). \end{aligned} \tag{1}$$

We shall assume that the functions $\varphi_i(t)$ are given and do not decrease on $[0; +\infty)$, are continuous, and $\varphi_i(0) = 0$.

The obtained class of functions is a compact** set in the space of continuous functions on Π with the uniform metric. Denote by $H_\varepsilon(W)$ its ε-entropy (for the definition see ([3,9])).

* The notation of the class W reflects the fact that it can be regarded as a topologized tensor product of the corresponding one-dimensional classes.

** By a compact set we shall everywhere mean a set that is completely bounded in C (not necessarily closed).

2. The ε-entropy of the compact set W admits a simple upper estimate.

Theorem 1. *For each class W there exist positive constants C_1, C_2, depending only on k, such that*

$$H_{C_1\varepsilon}(W) \leq C_2 \prod_{i=1}^{n} \frac{1}{\psi_i(\varepsilon)} + O\left(\log_2 \frac{1}{\varepsilon}\right),$$

where $\psi_i(t) = \sup\{\alpha : \psi(t) \leq t\}$; if, however,

$$\lim_{t\to 0} \frac{\psi_i(t)}{t^k} = 0,$$

then we put $\psi_i(t) \equiv 1$.

For the proof we use the following lemma, which is a slight generalization of Whitney' s result (see ([10])).

Lemma. *Let a bounded function $f(x)$ be given on the n-dimensional parallelepiped*

$$\Pi = \prod_{i=1}^{n} [a_i, b_i],$$

and let

$$E_l(f; \Pi) = \inf_{P_l(x)} \sup_{x\in\Pi} |f(x) - P_l(x)|,$$

where $P_l(x)$ is a polynomial whose degree in the i-th variable is not higher than l_i $(l = (l_i)_{i=1}^n)$. Then there exists a constant A_l (not depending on the function f) such that

$$E_l(f; \Pi) \leq A_l \sum_{i=1}^{n} \omega_{l_i+1}^{(i)} \left(f; \frac{b_i - a_i}{l_i + 1}\right).$$

3. Passing to lower estimates of $H_\varepsilon(W)$, let us note at once that, in order to obtain an effective result, it is necessary to impose certain restrictions on the functions $\varphi_i(t)$. Indeed, as follows from the results of ([8]), if a function $\varphi(t)$ satisfying the conditions

$$\varphi(t) \geq Ct^k, \qquad \varphi(t) \leq C_1 t^k, \qquad 0 \leq t \leq b - a$$

(the second of the inequalities being satisfied at least in some neighborhood of zero), then the class $(\Delta^k(\varphi), [a, b], M)$, defined by this function, possesses $(k-1)$ derivatives which satisfy a Lipschitz condition with constant C_1. Therefore, if we want to obtain a lower estimate coinciding with the upper estimate, we must take the majorants in some sense as the smallest possible; namely, if $W = (\Delta^k(\varphi), [a, b], M)$, then by the smallest majorant we shall mean

$$\overline{\varphi}(t) = \sup_{f \in W} \omega_k(f; t).$$

It is clear that $\overline{\varphi}(t) \leq \varphi(t)$. But from the definition itself it is evident that $\overline{\varphi}(t)$ must possess certain properties of a k-th modulus of smoothness. For $k = 1$ such characteristic properties are known (see, for example, ([9])). Relying on this, it is not difficult to obtain the following estimate of $H_\varepsilon(W)$ for the class $W = (\Delta^1(\varphi_i), \Pi, M)$, where all φ_i are moduli of continuity:

$$H_\varepsilon(W) \geq C_1 \prod_{i=1}^{n} \frac{1}{\psi_i(C\varepsilon)}$$

(compare with Theorem 8 of ([2])).

For moduli of smoothness of order $k \geqslant 2$, we shall restrict ourselves to the most frequently occurring case of power majorants.

Let

$$W = \left(\prod_{i=1}^{n} \otimes \Delta^{k_i}(t^{\beta_i}), \Pi, M \right) \quad (\beta_i \leqslant k_i),$$

where

$$\Pi = \prod_{i=1}^{n} [a_i, b_i].$$

Then

$$H_\varepsilon(W) \geqslant C\varepsilon^{-\sum_{i=1}{}^{n} \frac{1}{\beta_i}}$$

(cf. the corresponding result of paper ([5])).

Comparing the estimate obtained with the result of Theorem 1, we obtain the following assertion.

Theorem 2. The exact order of growth of the ε-entropy of the compact set

$$W=\left(\prod_{i=1}^{n}\otimes\Delta^{k_i}(t^{\beta_i}),\Pi,M\right)\quad(\beta_i\leqslant k_i)$$

is equal to $\varepsilon^{-\sum_{i=1}{}^{n}\frac{1}{\beta_i}}$, i.e.,

$$H_\varepsilon(W)\asymp\varepsilon^{-\sum_{i=1}{}^{n}\frac{1}{\beta_i}}. \tag{2}$$

Using the results of papers ([8],[1]), one can give a more convenient description of the classes

$$W=\left(\prod_{i=1}^{n}\otimes\Delta^{k_i}(t^{\beta_i}),\Pi,M\right)\quad(\beta_i\leqslant k_i).$$

Namely, it follows from these papers that if $\omega_k(f;t)\leqslant t^\beta$ $(\beta\leqslant k)$, then:

a) for $\beta=k$, $f(x)$ has a derivative of order $s=\beta-1$ satisfying the condition

$$\left|\Delta_h^1 f^{(s)}(t)\right|\leqslant h;$$

b) for fractional β, $f(x)$ has $s=[\beta]$ derivatives, and

$$\left|\Delta_h^1 f^{(s)}(t)\right|\leqslant C(\beta)h^{\beta-s};$$

c) for integral $\beta\neq k$, $f(x)$ has a derivative of order $s=\beta-1$, and

$$\left|\Delta_h^2 f^{(s)}(t)\right|\leqslant C(\beta)h.$$

Hence it follows that the class W consists of functions having, in each variable, the corresponding number of derivatives which, depending on the relations between β_i and k_i, satisfy one of the inequalities in items a)—c).

In particular, for $n=1$, $\beta=k-\alpha$ $(0\leqslant\alpha\leqslant1)$, it follows from (2) that the ε-entropy of the corresponding compact set (which consists of uniformly bounded functions differentiable $k-1$ times, with the $(k-1)$-st derivative satisfying the Hölder condition of order $1-\alpha$) has order $\varepsilon^{-1/\beta}$. This result was obtained earlier by A. N. Kolmogorov ([3]).

If $n=1$, $\beta=k-1$, then the ε-entropy of the corresponding compact set, consisting of functions differentiable $k-2$ times whose $(k-2)$-nd derivative is quasi-smooth, has order $\varepsilon^{-1/\beta}$. For $k=2$, this result was obtained earlier by B. D. Kotlyar (see ([6])).

Remark 1. Consider the class

$$W=\left(\prod_{i=1}^{n}\otimes\Delta^{k_i}(t_i^{\beta_i}),\Pi,M\right),$$

where $k_i=s$, $\beta=s-\alpha$ $(0\leqslant\alpha\leqslant 1)$. Then the corresponding estimate has the form

$$H_\varepsilon(W)\asymp\varepsilon^{-\frac{n}{s-\alpha}}.$$

It is interesting to compare this result with the result obtained by A. N. Kolmogorov. In ([3]) it is proved that

$$H_\varepsilon(W_1')\asymp\varepsilon^{-\frac{n}{s-\alpha}},$$

where W_1' is the class of functions defined on Π and possessing mixed derivatives up to order $(s-1)$ inclusive, satisfying a Hölder condition of order $1-\alpha$ $(0<\alpha<1)$. Since in the definition of W the existence of derivatives of order $(s-1)$ is required only with respect to each variable separately (they satisfy a Hölder condition of order $1-\alpha$), the compact set W, generally speaking, is broader than W_1'. However, if these classes consist of periodic functions, then, by virtue of Montel' s result (see ([7])), the indicated classes coincide (see also ([9])). In the nonperiodic case, however, it has been proved only that the corresponding mixed derivatives exist in any closed domain G lying strictly inside the parallelepiped. But, apparently, in reality the indicated classes coincide also in the nonperiodic case.

Remark 2. Estimate (2) makes it possible to strengthen A. G. Vitushkin' s result on the impossibility of representing a function of several variables by functions of a smaller number of variables. Namely, repeating A. N. Kolmogorov' s reasoning (see ([4])), one can prove this theorem for the case when the functions composing the superposition have different differentiability properties with respect to different variables.

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REFERENCES

[1] Yu. A. Brudnyi, I. E. Gopengauz, DAN, **141**, No. 6 (1961).
[2] Yu. A. Brudnyi, A. F. Timan, DAN, **128**, No. 6 (1959).
[3] A. N. Kolmogorov, DAN, **108**, No. 3 (1956).
[4] A. N. Kolmogorov, UMN, **10**, no. 1, 192 (1955).
[5] A. N. Kolmogorov, V. M. Tikhomirov, UMN, **14**, no. 2, 8 (1959).
[6] B. D. Kotlyar, UMN, **18** (1963).
[7] P. Montel, Bull. Soc. Math. France, **46**, 191 (1918).
[8] D. A. Raikov, DAN, **24**, 652 (1939).

[9] A. F. Timan, *Theory of Approximation of Functions of a Real Variable*, 1960.
[10] H. Whitney, Proc. Am. Math. Soc., **10**, No. 3, 480 (1959).

Note: Figure translations are in progress. See original paper for figures.