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Abstract

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ESTIMATES OF SOLUTIONS OF GENERAL BOUNDARY-VALUE PROBLEMS FOR ELLIPTIC SYSTEMS

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In a domain Ω of the n-dimensional space E_n, consider the system with real coefficients

$$\sum_{j=1}^{m} l_{ij}\left(x, \frac{\partial}{\partial x}\right) u_j(x) = f_i(x) \qquad (i = 1, \dots, m), \tag{1}$$

elliptic in the sense of Douglis–Nirenberg [1]. This means that the order of the operator l_{ij} may depend on both indices, but there exist integers s_i and t_i $(i = 1, 2, \dots, m)$ such that: 1) the order of l_{ij} does not exceed $s_i + t_j$, and, when $s_i + t_j < 0$, $l_{ij} = 0$; and 2) the polynomial

$$L(x, \xi) = \det\{l^0_{ij}(x, \xi)\},$$

where l^0_{ij} is the principal part of the operator l_{ij}, containing terms of order exactly $s_i + t_j$, is nonzero for no real $\xi = (\xi_1, \dots, \xi_n) \neq 0$.

Obviously, $L(x, \xi)$ is a homogeneous polynomial in ξ of degree

$$\sum_{i=1}^{m} (s_i + t_i).$$

From the ellipticity condition it follows that

$$\sum_{i=1}^{m} (s_i + t_i) = 2r,$$

where r is a nonnegative integer. We shall assume that $r > 0$.

By adding, if necessary, a constant to all s_i and subtracting it from all t_i, one can arrange that $\max_i s_i = 0$. Then necessarily $\min_i t_i \geq 0$.

By $L_{ij}(x, \xi)$ we denote the algebraic cofactor of the element $l^0_{ij}(x, \xi)$; obviously, $L_{ij}(x, \xi)$ is a homogeneous polynomial of degree $2r - s_i - t_j$, and if $2r - s_i - t_j < 0$, then $L_{ij} = 0$.

Systems elliptic in the sense of Petrovskii are a special case of systems elliptic in the sense of Douglis–Nirenberg. An example of a system elliptic in the sense of Douglis–Nirenberg, but not in the sense of Petrovskii, is the linearized stationary system of the Stokes equations. It contains only the highest-order terms.

We shall consider a solution of system (1) satisfying on the boundary S of the domain Ω the following boundary conditions:

$$\sum_{j=1}^{m} B_{qj}\left(x, \frac{\partial}{\partial x}\right) u_j(x)\bigg|_S = \Phi_q(x) \qquad (q = 1, \dots, r). \tag{2}$$

Suppose that there exist integers σ_q $(q = 1, \dots, r)$ such that:

1. The order of B_{qj} does not exceed $\sigma_q + t_j$, and if $\sigma_q + t_j < 0$, then $B_{qj} = 0$.
2. Let
$$C_{qk}(x, \xi) = \sum_{j=1}^{m} B^0_{qj} L_{kj},$$
where $B^0_{qj}(x, \xi)$ is the principal part of B_{qj}, containing terms of degree $\sigma_q + t_j$, $\nu = (\nu_1, \dots, \nu_n)$ is the vector of the inner normal to S at the point x, and $\xi = (\xi_1, \dots, \xi_n)$ is a vector lying in the tangent plane to S at the point x, i.e. such that
$$\sum_{k=1}^{n} \xi_k \nu_k = 0.$$
The polynomials C_{qk} (that is, ultimately, the polynomials B^0_{qj}) must satisfy the condition:

Complementing condition. At no point $x \in S$ and for no nonzero ζ do there exist such d_q, not all vanishing simultaneously, for which the polynomials in the complex variable τ

$$\sum_{q=1}^{r} d_q C_{qk}(x, \zeta + \tau\nu) \qquad (k = 1, \dots, m)$$

would be simultaneously divisible by the polynomial

$$M^+(x, \zeta, \tau) = \prod_{s=1}^{r} (\tau - \tau_s^+(x, \zeta)),$$

where $\tau_s^+(x, \zeta)$ are the roots of the polynomial $L(x, \zeta + \tau\nu)$, considered as a function of τ, that lie in the upper half-plane of the complex τ-plane and are numbered taking account of their multiplicities.

This condition is analogous to the condition formulated in paper ([2]) for the case of a single elliptic equation of order $2r$. It replaces the well-known condition of

Ya. B. Lopatinskii, formulated in paper ([3]) as applied to general boundary-value problems for elliptic systems in the sense of Petrovskii. In that paper general boundary-value problems for such systems were first considered, an explicit representation of their solutions in the half-space was obtained, and boundary-value problems in domains were reduced to integral equations*.

Under natural smoothness assumptions on the coefficients of the operators l_{ij} and B_{qj} and on the boundary S, the following theorem holds:

Theorem. For the solution of problem (1), (2) the estimates

$$\sum_{i=1}^{m} \|u_i\|_{C^{l+t_i}(\Omega)} \le C_1 \left[\sum_{q=1}^{r} \|\Phi_q\|_{C^{l-\sigma_q}(S)} + \sum_{i=1}^{m} \|f_i\|_{C^{l-s_i}(\Omega)} + \sum_{i=1}^{m} \max |u_i| \right],$$

where $l > \max(0, \sigma_1, \dots, \sigma_r)$, $l = [l] + \alpha$, $0 < \alpha < 1$, and

$$\sum_{i=1}^{m} \|u_i\|_{W_p^{l+t_i}(\Omega)} \le C_2 \left[\sum_{q=1}^{r} \|\Phi_q\|_{W_p^{l-\sigma_q-1/p}(S)} + \sum_{i=1}^{m} \|f_i\|_{W_p^{l-s_i}(\Omega)} + \sum_{i=1}^{m} \|u_i\|_{L_p(\Omega)} \right]$$

hold, where l is a nonnegative integer and $l > 1/p + \max(0, \sigma_1, \dots, \sigma_r)$; Ω is a bounded domain**.

The proof of this theorem is carried out by means of a method connected with decomposing the domain Ω into subdomains, straightening the boundary, and subsequently considering the solution of problem (1), (2) in each subdomain. The idea of this method belongs, as is known, to Schauder. It allows one to reduce the proof of the theorem to an estimate of solutions of elliptic systems with constant coefficients, containing only the principal terms, in the whole space E_n, as well as of solutions of boundary-value problems for such systems in a half-space:

$$\sum_{j=1}^{m} l_{ij}^{0} \left(\frac{\partial}{\partial x} \right) u_j = f_i(x), \qquad (x_n \ge 0),$$

$$\sum_{j=1}^{m} B_{qj}^{0} \left(\frac{\partial}{\partial x} \right) u_j \bigg|_{x_n=0} = \Phi_q(x_1, \dots, x_{n-1}).$$

* M. Z. Solomyak drew my attention to the equivalence of the complementing condition given above and the Lopatinskii condition in the case of elliptic systems in the sense of Petrovskii.

** $C^{r+\alpha}$ is the space of functions whose derivatives of order r satisfy the Hölder condition with exponent α; W_p^l for integer l are the spaces of S. L. Sobolev, and for fractional l the spaces of L. N. Slobodetskii ([6]).

The solution of this problem can be represented in the form

$$u_j = \sum_{k=1}^{m} L_{kj}\left(\frac{\partial}{\partial x}\right) v_k(x),$$

where the functions v_k satisfy the equations

$$L\left(\frac{\partial}{\partial x}\right) v_k = f_k \qquad \left(L(\xi) = \mathrm{Det}\{l_{ij}^0(\xi)\}\right)$$

and the boundary conditions

$$\sum_{k=1}^{m} C_{qk}\left(\frac{\partial}{\partial x}\right) v_k\bigg|_{x_n=0} = \Phi_q \qquad \left(C_{qk}(\xi) = \sum_{j=1}^{m} B_{qj}^0(\xi) L_{kj}(\xi)\right).$$

Therefore it is clear that the derivatives of the functions u_j can be expressed in terms of f_k and Φ_q with the aid of the fundamental solution of the equation $Lu = 0$ and "Poisson kernels" analogous to those constructed in ([2]).

Estimates of the solution of problem (1), (2) can also be obtained in other fractional norms, for example in the norms B_p^l and H_p^l.

It should be noted that estimates of solutions of the system (1) in Hölder norms inside the domain Ω were obtained in ([1]). For the case of a single elliptic equation, the theorem formulated by us was proved in ([2,4]). Estimates in the norms W_p^l for systems elliptic in the sense of Petrovskii were announced in ([5]); estimates in the norms W_2^l (where l is not necessarily an integer) were formulated in ([6]) and proved in ([7]). In place of Ya. B. Lopatinskii' s condition, in ([5−7]) a somewhat stronger restriction is imposed on the principal parts of the operators entering into the boundary conditions. In ([8]), boundary-value problems are considered for systems elliptic in the sense of Petrovskii with integro-differential boundary conditions, and a theorem is formulated on the estimate of their solutions in the norm W_2^l, where l is not necessarily an integer.

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Note: Figure translations are in progress. See original paper for figures.