

On the Arrangement of Charges at the Vertices of the Unit (n) -Dimensional Cube*

1963

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196301.50293>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

Cybernetics and Control Theory

B. S. Zilberman

On the Arrangement of Charges at the Vertices of the Unit n -Dimensional Cube*

(Presented by Academician P. S. Novikov on 11 X 1962)

We consider the set E_n of all vertices of the unit n -dimensional cube, i.e., the set of binary strings $(\sigma_1, \dots, \sigma_n)$ of length n . The set E_n becomes a metric space if, for any pair of its elements $A = (\alpha_1, \dots, \alpha_n)$ and $B = (\beta_1, \dots, \beta_n)$, the distance is defined by

$$\rho(A, B) = |\alpha_1 - \beta_1| + |\alpha_2 - \beta_2| + \dots + |\alpha_n - \beta_n| \quad (1).$$

To each set of vertices $M = \{A_1, \dots, A_m\}$ ($m \geq 2$) there is assigned a number $H(M)$, called the energy of the given set:

$$H(M) = \sum_{1 \leq i < j \leq m} \frac{1}{\rho(A_i, A_j)}.$$

The problem is, among all sets of m vertices of the unit n -dimensional cube ($2 \leq m \leq 2^n$), to find a set with minimal energy.

If the number $H(M)$ is taken as the value of the energy of a system of equally charged particles located at the vertices A_1, \dots, A_m , then this problem can be given a physical meaning: to place m equally charged particles at the vertices of the unit n -dimensional cube so that the resulting system of charges is stable**.

The problem of arranging charges at the vertices of the unit n -dimensional cube arose in connection with the problem of constructing optimal self-correcting codes⁽¹⁾. For this it is necessary to consider the coding problem in a somewhat different formulation. Namely: to find such a system of m code points in the unit n -dimensional cube that the probability of detecting and correcting an error (for a given noise source) is greatest. There is an opinion that every solution of the charge problem will be a solution of the coding problem. In that case it will be possible to construct optimal codes using physical principles.

A complete solution of the charge problem has proved difficult. The author investigated the case when $m = 2^{n-1}$. In the present note*** it is shown that in this case there exist only two solutions, namely, the sets whose characteristic functions are the linear functions of n variables $x_1 + \dots + x_n \pmod{2}$ and $x_1 + \dots + x_{n-1} + 1 \pmod{2}$. Consequently, for this special case the supposition stated above has been fully confirmed.

Let

$$g(n) = C_n^1/1 + C_n^2/2 + \dots + C_n^n/n, \quad \overline{M} = E_n \setminus M.$$

It is easily proved

Theorem 1.

$$H(\overline{M}) - H(M) = (2^{n-1} - m)g(n). \quad (1)$$

Theorem 1 makes it possible to restrict ourselves to consideration of the case $m \leq 2^{n-1}$. Number all points of the cube $E_n = \{A_1, \dots, A_{2^n}\}$ arbitrarily. Let us partition the cube into two sets M and $\overline{M} = E_n \setminus M$. Denote by

* The work was reported at the seminar on discrete analysis at Moscow State University named after M. V. Lomonosov in the spring of 1961.

** This problem was posed and suggested to the author by S. V. Yablonskii as a thesis topic.

*** The present note is a development of the 1960 thesis.

$a_i^{(s)}$ be the number of points lying at distance s from the point A_i and belonging to the same set (M or \overline{M}) as A_i . With the aid of these numbers one obtains a simple expression for the energy

$$H(M) = \frac{1}{4} \sum_{s=1}^n \frac{1}{s} \sum_{i=1}^{2^n} a_i^{(s)} - \frac{1}{2} (2^{n-1} - m)g(n). \quad (2)$$

Lemma 1. The numbers $a_i^{(s)}$ satisfy the relations:

$$\begin{aligned} & \sum_{i=1}^{2^n} a_i^{(1)} a_i^{(s)} + \sum_{i=1}^{2^n} (n - a_i^{(1)}) (C_n^s - a_i^{(s)}) \\ &= (n - s + 1) \sum_{i=1}^{2^n} a_i^{(s-1)} + (s + 1) \sum_{i=1}^{2^n} a_i^{(s+1)} \quad (3) \\ & (s = 1, \dots, n - 1), \quad a_i^{(0)} = 1. \end{aligned}$$

The numbers $a_i^{(s)}$, as is easy to see, also satisfy the conditions

$$\sum_{i=1}^{2^n} \sum_{s=1}^n a_i^{(s)} = m(m - 1) + (2^n - m - 1)(2^n - m),$$

where m is the cardinality of M . In what follows we everywhere assume that $m = 2^{n-1}$.

Consider the system of equations in the $a_i^{(s)}$ ($i = 1, \dots, 2^n$; $s = 1, \dots, n$)

$$\sum_{i=1}^{2^n} a_i^{(1)} a_i^{(s)} + \sum_{i=1}^{2^n} (n - a_i^{(1)}) (C_n^s - a_i^{(s)}) - (n - s + 1) \sum_{i=1}^{2^n} a_i^{(s-1)} - (s + 1) \sum_{i=1}^{2^n} a_i^{(s+1)} = 0 \quad (s = 1, \dots, n - 1); \quad (4)$$

$$\sum_{i=1}^{2^n} \sum_{s=1}^n a_i^{(s)} - 2^n (2^{n-1} - 1) = 0.$$

Obviously, every partition of the cube E_n into two sets of equal cardinality gives rise to a solution of system (4).

We shall call a solution of system (4) homogeneous if it has $a_i^{(1)} = a_j^{(1)} = a^{(1)}$ ($i, j = 1, \dots, 2^n$). A set M of vertices of the cube E_n will be called homogeneous if the corresponding partition of the cube gives rise to a homogeneous solution. Let us consider an important class of homogeneous sets.

A parity counter of order k , E_n^k ($k = 0, 1, \dots, n - 1$), is a set whose characteristic function is $f(x_1, \dots, x_n) = x_{j_1} + \dots + x_{j_{n-k}} + c \pmod{2}$, $c \in \{0, 1\}$.

It is easy to see that for the parity counter E_n^k

$$a_i^{(s)} = a_j^{(s)} = p_k^{(s)} \quad (i, j = 1, \dots, 2^n; s = 1, \dots, n),$$

where

$$p_k^{(s)} = C_k^s + C_k^{s-2} C_{n-k}^2 + C_k^{s-4} C_{n-k}^4 + \dots \quad (5)$$

In particular, $a_i^{(1)} = a_j^{(1)} = k$; consequently, the set E^k is homogeneous.

Denote by $a^{(s)}$ the quantity $2^{-n} \sum_{i=1}^{2^n} a_i^{(s)}$.

Lemma 2. For a homogeneous solution one has $a^{(s)} = p_k^{(s)}$ ($s = 1, \dots, n$).

Proof. For a homogeneous solution, system (4) has the form

$$(s + 1)a^{(s+1)} = 2a^{(1)}a^{(s)} - na^{(s)} - (n - s + 1)a^{(s-1)} - C_n^s a^{(1)} + nC_n^s \quad (s = 1, \dots, n - 1),$$

$$a^{(1)} + a^{(2)} + \dots + a^{(n)} = 2^{n-1} - 1. \quad (6)$$

It is clear from this that $a^{(s+1)}$ is expressed as a polynomial in $a^{(1)}$ of degree $s+1$. Substituting, in the last of equations (6), their expressions through $a^{(1)}$ in place of $a^{(2)}, \dots, a^{(n)}$, we obtain a polynomial of degree n in $a^{(1)}$. Consequently, system (6) has n solutions. But the n distinct solutions of system (6) generate the sets E_n^k ($k = 0, 1, \dots, n-1$), and therefore $a^{(s)} = p_k^{(s)}$ ($s = 1, \dots, n$). The lemma is proved.

Corollary. The energy of a homogeneous solution, defined as

$$\frac{1}{4} \sum_{s=1}^n \frac{1}{s} \sum_{i=1}^{2^n} a_i^{(s)},$$

is equal to the energy of one of the parity counters.

Lemma 3. Among all parity counters, the set E_n^0 has the minimal energy.

Proof. By virtue of (5), the energy of the set E_n^k is equal to

$$H(E_n^k) = 2^{n-2} \sum_{s=1}^n \frac{1}{s} p_k^{(s)}.$$

Let

$$P_k(x) = \frac{(1+x)^{n-k} + (1-x)^{n-k}}{2} (1+x)^k - 1 \quad \text{and} \quad Q_k(x) = \int_0^x \frac{P_k(x)}{x} dx.$$

Obviously,

$$P_k(x) = \sum_{s=1}^n p_k^{(s)} x^s \quad \text{and} \quad Q_k(x) = \sum_{s=1}^n \frac{p_k^{(s)}}{s} x^s.$$

Further,

$$H(E_n^k) = 2^{n-2} Q_k(1) = 2^{n-2} \int_0^1 \frac{(1+x)^n + (1+x)^n \left(\frac{1-x}{1+x}\right)^{n-k} - 2}{2x} dx.$$

Hence it follows immediately that

$$H(E_n^0) < H(E_n^1) < \dots < H(E_n^{n-1}). \quad (7)$$

The lemma is proved.

Lemma 4. For any solution of system (4): 1) $a^{(2)} \geq 0$; 2) $0 \leq a^{(1)} \leq n$, if $a^{(2)} \leq C_n^2$.

Proof. Equations (4) for $s = 1$ give

$$2 \sum_{i=1}^{2^n} a_i^{(2)} = 2 \sum_{i=1}^{2^n} a_i^{(1)2} - 2n \sum_{i=1}^{2^n} a_i^{(1)} + n^2 \cdot 2^n - n \cdot 2^n.$$

We shall have

$$a^{(2)} \geq a^{(1)2} - na^{(1)} + C_n^2 \quad (\text{for } n \geq 2),$$

and also $0 \leq a^{(1)} \leq n$, if $a^{(2)} \leq C_n^2$. The lemma is proved.

Denote by F_s ($s = 1, 2, \dots, n-1$), F_0 the left-hand sides of equations (4), and by G_r the quantity

$$\sum_{i=1}^{2^n} a_i^{(r)} \quad (r = 1, \dots, n).$$

We shall seek the minimum of the function $H' = 4H = \sum_{s=1}^n \frac{1}{s} \sum_{i=1}^{2^n} a_i^{(s)}$ on the manifold defined by the equations $F_0 = 0, F_1 = 0, \dots, F_{n-1} = 0$ and the boundary conditions $G_1 \geq 0, G_3 \geq 0, G_4 \geq 0, \dots, G_n \geq 0$. In view of these boundary conditions and Lemma 4, the function H' is bounded from below and hence, being continuous and defined on a closed manifold, attains its minimum value.

Lemma 5. For $n > 3$, the minimum of the function H' is attained at a homogeneous solution.

Proof. We apply the method of Lagrange multipliers. Consider the function

$$\Phi = H' + \lambda_0 F_0 + \sum_{s=1}^{n-1} \lambda_s F_s + \sum_{r=1, r \neq 2}^n \mu_r \sigma_r G_r,$$

where $\lambda_0, \lambda_1, \dots, \lambda_{n-1}, \mu_1, \mu_3, \mu_4, \dots, \mu_n$ are Lagrange multipliers, $\sigma_r \in \{0, 1\}$. (σ_r is introduced so as not to consider separately the different cases of the position of a stationary point: to each set $(\sigma_1, \sigma_3, \sigma_4, \dots, \sigma_n)$, where not all σ_r are equal to 0, there corresponds a certain position on the boundary; the set $(0, 0, 0, \dots, 0)$ corresponds to an interior point.) We exclude from consideration the set $(1, 1, 1, \dots, 1)$, since in this case the energy is obviously not minimal for $n > 3$. Indeed, then

$$\sum_{i=1}^{2^n} a_i^{(2)} = 2^n(2^{n-1} - 1)$$

and

$$H' = \frac{1}{2} \sum_{i=1}^{2^n} a_i^{(2)} > 4H(E_n^0) \quad (n > 3).$$

Let us write the Lagrange equations for the function Φ :

$$\begin{aligned} \partial\Phi/\partial a_i^{(1)} &= 1 + \lambda_0 + \sigma_1\mu_1 + 2\lambda_1(2a_i^{(1)} - n) + \lambda_2(2a_i^{(2)} - C_n^2 - n + 1) \\ &\quad + \lambda_3(2a_i^{(3)} - C_n^3) + \dots + \lambda_{n-1}(2a_i^{(n-1)} - C_n^{n-1}) = 0, \\ \partial\Phi/\partial a_i^{(2)} &= 1/2 + \lambda_0 - 2\lambda_1 + \lambda_2(2a_i^{(1)} - n) - \lambda_3(n - 2) = 0, \\ \partial\Phi/\partial a_i^{(3)} &= 1/3 + \lambda_0 + \sigma_3\mu_3 - 3\lambda_2 + \lambda_3(2a_i^{(1)} - n) - \lambda_4(n - 3) = 0, \\ &\dots \dots \dots \\ \partial\Phi/\partial a_i^{(l)} &= 1/l + \lambda_0 + \sigma_l\mu_l - l\lambda_{l-1} + \lambda_l(2a_i^{(1)} - n) - \lambda_{l+1}(n - l) = 0, \\ &\dots \dots \dots \\ \partial\Phi/\partial a_i^{(n-1)} &= 1/(n - 1) + \lambda_0 + \sigma_{n-1}\mu_{n-1} - (n - 1)\lambda_{n-2} + \lambda_{n-1}(2a_i^{(1)} - n) = 0, \\ \partial\Phi/\partial a_i^{(n)} &= 1/n + \lambda_0 + \sigma_n\mu_n - n\lambda_{n-1} = 0 \\ &\quad (i = 1, \dots, 2^n). \end{aligned} \tag{8}$$

From equations (8) and the condition $\sigma_l = 0$ ($l \neq 2$) it follows that $a_i^{(1)} = a_j^{(1)}$ ($i, j = 1, \dots, 2^n$). The lemma is proved.

Comparing Lemmas 2, 3, and 5, and noting that the proposition stated below is obvious for $n \leq 3$, we obtain Theorem 2.

Theorem 2. *The minimum energy of a set of cardinality 2^{n-1} is attained on the parity counter E_n^0 and is equal to*

$$H(E_n^0) = 2^{n-2} (C_n^2/2 + C_n^4/4 + \dots).$$

The uniqueness of the set with minimal energy follows from the uniqueness of the set for which $a_i^{(1)} = 0$ ($i = 1, \dots, 2^n$).

In an analogous way, using inequalities (7) and the second assertion of Lemma 4, one could show that the maximum of the energy is attained on the parity counter E_n^{n-1} , i.e., on the set whose characteristic function is equal to x_j ($j = 1, \dots, n$).

Moscow Electromechanical Institute

Received
14 IX 1962

REFERENCES

1. R. W. Hamming, in: *Codes with Error Detection and Correction*, IL, 1956, p. 7.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.