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Abstract

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Yu. A. Ryabov

Some Asymptotic Properties of Linear Systems with a Small Time Delay

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1. In the paper ([1]), in considering linear differential equations with a retarded argument (for which time t is taken), it was shown that, by means of the small-parameter method, it is possible to construct, if the delay does not exceed a certain limit depending on the coefficients of the equations, a certain family of two-sided particular solutions of such equations. These solutions (let us call them special *), which are uniquely determined when only the initial values of the unknown functions are prescribed and which satisfy certain linear differential equations without delay, make it possible to approximate sufficiently accurately, in any case beginning from some instant of time after the initial one and on some finite interval, all possible other solutions of the original equations, determined by one or another initial function. However, a stronger result can be proved, namely, that any solution of linear equations with bounded coefficients and with a delay not exceeding a certain limit is not only approximated by special solutions on a finite interval of variation of t, but also approaches asymptotically (the faster, the smaller the delay) one of the special solutions.

Consider the equation

$$\dot{x}(t) = px(t) + qx(t-h), \tag{1}$$

where p, q, h are bounded, piecewise-continuous functions of t for $-\infty < t < \infty$.

Theorem 1. Let $x = \psi(t)$ be a solution of equation (1), corresponding to an initial function $\varphi(t)$, integrable and bounded in modulus. Then, if the upper bound h_0 of the delay h does not exceed a certain limit independent of $\varphi(t)$, there exists such a special solution $\tilde{x}_\psi(t)$ of this equation that

$$\lim_{t\to\infty} [\psi(t) - \tilde{x}_\psi(t)] = 0. \tag{2}$$

The scheme of the proof is as follows. Let a sequence of instants $t_k = kh_0,\ k = 2, 3, \dots$, be fixed, and let the solution under investigation $x = \psi(t)$ take at these instants the values $x_{k0} = \psi(t_k),\ k = 2, 3, \dots$. Consider the sequence of special solutions corresponding to the initial values $x(t_k) = x_{k0}$, denoting these solutions by $\tilde{x}(t, t_k)$, $k = 2, 3, \ldots$. In accordance with what was set forth in ([1]), the solution $x = \psi(t)$ is represented in a neighborhood of any instant $t = t_k,\ k \geqslant 2$, in the form

$$\psi(t) = \tilde{x}(t, t_k) + \Delta_k(t), \tag{3}$$

where $\Delta_k(t)$ is the smaller the larger k is, if h_0 satisfies the inequality

$$a = ch_0 e \cdot e^{1/e} < 1, \tag{3*}$$

where $c = \sup(|p| + |q|),\ -\infty < t < \infty$. One can obtain the following estimate for $\Delta_k(t)$, $k \geqslant 2$:

$$|\Delta_k(t)| < Aa^k, \qquad t \in [t_k - h_0,\ t_k + h_0], \tag{4}$$

* This name seems more appropriate than the name "asymptotic" solutions used in ([1]).

where A is some constant depending on the coefficients of the equation, bounded as $h_0 \to 0$ and proportional to $\sup|\varphi(t)|,\ -h_0 \leq t \leq 0$. With the aid of this estimate, and also of the estimate of the growth of special solutions

$$|\tilde{x}(t, t_k)| \leq |\psi(t_k)| \frac{e^{-c\lambda h_0}}{1 - c\lambda h_0} e^{c\lambda|t - t_k|}, \tag{5}$$

where λ is the smallest real root of the equation $\lambda = e^{c\lambda h_0}$, it is possible to prove that, in any case, if

$$\bar{\alpha} = \alpha e^{c\lambda h_0} < 1, \tag{5*}$$

then the sequence of special solutions $\{\tilde{x}(t, t_k)\}$ converges to some special solution $\tilde{x}_\psi(t)$, and moreover not only for any fixed $t = t_*$ do we have $\lim_{k\to\infty} \tilde{x}(t_*, t_k) = \tilde{x}_\psi(t_*)$, but, in addition,

$$\left|\tilde{x}(t, t_k) - \tilde{x}_\psi(t)\right| < \bar{A}\frac{\bar{\alpha}^k}{1 - \bar{\alpha}}, \qquad \bar{A} = \frac{A}{1 - c\lambda h_0}. \tag{6}$$

As k increases, ever greater closeness of $\tilde{x}(t, t_k)$ and $\tilde{x}_\psi(t)$ is achieved throughout the interval $[0, t_{k+1}]$, whose length grows together with k. With the aid of (6) and (4) we obtain that, for a given arbitrarily small $\varepsilon > 0$, there is an N such that for all $t \geq Nh_0$ we have $|\psi(t) - \tilde{x}_\psi(t)| < \varepsilon$, whence the validity of the theorem follows. The quantitative estimate of the rate of convergence between $\psi(t)$ and $\tilde{x}_\psi(t)^*$ as t increases is as follows:

$$\left|\psi(t)-\tilde{x}_{\psi}(t)\right|<B\alpha^{k}, \qquad kh_0 \leq t \leq (k+1)h_0, \tag{7}$$

where $B = A + \bar{A}/(1-\bar{\alpha})$, $k = 2, 3, ...$

2. If we choose any sequence of moments $t = \tau_k$, $k = 1, 2, 3, ...$, such that $\tau_k \to \infty$ as $k \to \infty$, then the sequence of special solutions found for the initial values $x(\tau_k) = \psi(\tau_k)$ also tends to the same limiting special solution $\tilde{x}_{\psi}(t)$ to which the sequence $\{\tilde{x}(t, t_k)\}$ considered above tends. This solution, defined as the limit of any sequence of special solutions $\{\tilde{x}(t, \tau_k)\}$, $\tau_k \to \infty$, $\tilde{x}(\tau_k, \tau_k) = \psi(\tau_k)$, will be called asymptotic with respect to the given solution $x = \psi(t)$ of the original equation (1).

With the aid of estimate (7) the following can be proved.

Theorem 2. *If h_0 does not exceed a certain limit, then any solution $x = \psi(t)$ of equation (1) and the corresponding asymptotic solution $\tilde{x}_{\psi}(t)$ have one and the same characteristic exponent.*

For positive and zero characteristic exponents this theorem follows directly from (2). For negative characteristic exponents it follows from (7) that, for values of h_0 not exceeding a certain limit, the decrease of $\psi(t)$ in modulus as t increases takes place much more slowly than the convergence to $\tilde{x}_{\psi}(t)$, which makes it possible to prove the validity of the theorem.

3. If one considers the infinite-dimensional set of solutions $\{x(t)\}$ of equation (1), corresponding to the collection of initial functions bounded in modulus by some number M, then one may conclude that, if h_0 does not exceed a certain limit, this set is, as it were, stratified into a one-dimensional family of subsets, all solutions of each of these subsets asymptotically approaching (the faster, the smaller h_0) one of the special solutions. Thus, all features of the set of integral curves of equation (1) caused by the different initial functions gradually disappear (the faster, the smaller h_0) as one moves away from the initial moment $t = 0$, while those properties are retained which are determined by the very structure of the equation.

* Provided that the asymptotic solution is different from the trivial zero solution.

These results touch upon the problem, considered by A. D. Myshkis ([2,3]), of splitting the solutions of linear equations with delay into two classes: solutions of the "principal class" and "rapidly decaying" solutions. In accordance with Theorems 1 and 2, the solutions of the principal class asymptotically approach, as $t \to \infty$, special solutions distinct from the trivial zero solution, while the "rapidly decaying" solutions approach the trivial zero solution $x \equiv 0$. The difference between a given solution of the principal class and the corresponding asymptotic solution is nothing other than a rapidly decaying solution of the original equation.

4. Theorems 1 and 2 make it possible to draw a conclusion about the equivalence of the asymptotic behavior of the special solutions of equation (1)

and of all possible other solutions of this equation. In particular, the following theorems are valid.

Theorem 3. *If the maximum value of the delay h_0 does not exceed a certain bound and if the nontrivial special solutions of equation (1) either tend to zero as $t \to \infty$ or remain bounded as $t \to \infty$, then all solutions of this equation corresponding to different initial functions also tend to zero in the first case and remain bounded as $t \to \infty$ in the second case.*

Theorem 4. *If the maximum value of the delay h_0 does not exceed a certain bound and if the nontrivial special solutions of equation (1) are unbounded as $t \to \infty$, then, with the exception of the subset of "rapidly decaying" solutions, all the other solutions of equation (1) are also unbounded as $t \to \infty$.*

Theorem 5. *If the subset of "rapidly decaying" solutions is excluded, then, under the condition that h_0 does not exceed a certain bound, the asymptotic behavior of all the remaining solutions of equation (1) (from the point of view of boundedness, unboundedness, and convergence to zero) does not depend on the initial functions.*

These same results are valid for general linear systems with delay and bounded coefficients, and also for nonhomogeneous linear systems, to which it is not difficult to extend the proof of Theorems 1 and 2. A theorem analogous to Theorem 5 was recently proved by A. Halanay ([4]) for linear systems with periodic coefficients.

Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.