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Abstract

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SOME PROPERTIES OF MEAN FUNCTIONS

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We shall consider questions connected with the averaging of a function given on a finite plane domain. The latter is assumed to be summable on the domain and continuous on some closed strip adjacent to the boundary. Averaging is understood in the usual sense, as the arithmetic mean of the function over a circle of a certain radius with center at the given point. The radius is variable and tends to zero on approaching the boundary. The averaging circle does not go beyond the domain bounded by simple closed curves. Repeated averagings of the function under consideration lead to a certain functional sequence; there is uniform convergence of this sequence to a function harmonic in the domain and continuous up to the boundary, taking on the boundary the same values as the given function.

We note that everything said for the case of two variables is extended in an obvious way to the multidimensional case.

Let D be a plane simply connected domain bounded by a simple closed curve Γ, and let the function $f(M)$ be summable on D and continuous on some closed strip adjacent to the boundary Γ.

Introduce the averaging operator A, which assigns to $f(M)$ the function $Af(M)$, by setting

$$Af(M) = f(M), \qquad M \in \Gamma;$$

$$Af(M) = \frac{1}{\operatorname{mes} C_{h(M)}} \iint_{C_{h(M)}} f(\xi, \eta)\, d\xi\, d\eta, \qquad M \in D,$$

where $C_{h(M)}$ denotes the circle of radius $h(M)$ with center at the point M, and $h(M)$ is a function continuous on $D + \Gamma$ satisfying the conditions
a) $h(M) > 0,\ M \in D$; b) $C_{h(M)} \subset D + \Gamma,\ M \in D + \Gamma$.

It is not difficult to see that the function $Af(M)$ is continuous on $D+\Gamma$. Therefore, for what follows it is sufficient to assume (and it is assumed) that the operator A acts in the space of functions continuous on $D+\Gamma$, with the possible exception of some simple closed curve lying in D, on which these functions may have discontinuities with finite jump.

At the same time the following properties of the averaging operator are obvious:

1. $A[f(M)+g(M)] = Af(M)+Ag(M),\ M \in D+\Gamma$.
2. $A[\nu \cdot f(M)] = \nu \cdot Af(M),\ M \in D+\Gamma$, where $\nu = \text{const}$.
3. $Af(M) \leq Ag(M),\ M \in D+\Gamma$, if $f(M) \leq g(M),\ M \in D+\Gamma$.
4. $Af(M) = f(M),\ M \in D+\Gamma$, if $f(M)$ is harmonic in D.

Denoting by $\|f\|$ the maximum of a function $f(M)$ continuous on $D+\Gamma$ for $M \in D+\Gamma$, and by n any positive integer, we further have:

5. The functions of the sequence $\{A^n f(M)\}$ are equicontinuous on any closed set contained in D.
6. If $f(M) = \nu,\ M \in \Gamma$, where $\nu = \text{const}$, and

$$\sup_{M \in D+\Gamma} f(M) > \nu,$$

then there exists a positive integer N such that

$$\|A^N f\| < \sup_{M \in D+\Gamma} f(M).$$

Let us note one more property of the operator, which, incidentally, can easily be obtained from those already indicated:

$$7. \quad |A^n f(M)| \leq \sup_{M \in D+\Gamma} |f(M)|, \quad M \in D+\Gamma.$$

Let the symbol $\rightrightarrows$ denote uniform convergence.

Theorem 1. *If the functions $f(M)$ and $\chi(M)$ are continuous on $D+\Gamma$, the function $\chi(M)$ is harmonic in D, and $f(M) = \chi(M)$, $M \in \Gamma$, then*

$$A^n f(M) \rightrightarrows \chi(M), \quad M \in D+\Gamma.$$

The proof of Theorem 1 can be obtained by using the above-listed properties of the averaging operator. Let us note here that it suffices to prove the theorem for a function that is continuous on $D+\Gamma$, nonnegative in D, and equal to zero on the boundary Γ.

Indeed, consider the function

$$\psi(M) = f(M) - \chi(M), \quad M \in D+\Gamma.$$

By the hypotheses of the theorem,

$$\psi(M) = 0, \quad M \in \Gamma.$$

By properties 1, 2, 4 of the operator,

$$A^n \psi(M) = A^n f(M) - \chi(M), \quad M \in D + \Gamma.$$

Therefore, if the relation

$$A^n \psi(M) \rightrightarrows 0, \quad M \in D + \Gamma, \tag{1}$$

is proved, then the assertion of the theorem will also be proved.

Define on $D + \Gamma$ the functions $\psi_+(M)$ and $\psi_-(M)$, setting

$$\psi_+(M) = \begin{cases} \psi(M), & \text{if } \psi(M) \geq 0, \\ 0, & \text{if } \psi(M) < 0; \end{cases} \qquad \psi_-(M) = \begin{cases} 0, & \text{if } \psi(M) > 0, \\ \psi(M), & \text{if } \psi(M) \leq 0. \end{cases}$$

We have, obviously,

$$\psi_-(M) \leq \psi(M) \leq \psi_+(M), \quad M \in D + \Gamma,$$

whence

$$A^n \psi_-(M) \leq A^n \psi(M) \leq A^n \psi_+(M), \quad M \in D + \Gamma.$$

If, for example, it is proved that

$$A^n \psi_+(M) \rightrightarrows 0, \quad M \in D + \Gamma, \tag{2}$$

then relation (1) will follow from the preceding inequalities, which also justifies the remark made above.

We next consider the particular case when

$$h(M) = \rho(M, \Gamma), \quad M \in D + \Gamma, \tag{3}$$

where $\rho(M, \Gamma)$ denotes the distance from the point $M \in D + \Gamma$ to the boundary Γ.

In the indicated case, the proof of the uniform convergence to zero of the sequence of functions $\{A^n \psi_+(M)\}$ can be obtained with the aid of considerations that make it possible, in a certain sense, to judge the rate of this convergence.

Indeed, we have $\|\psi_+\| \geq 0$. If $\|\psi_+\| = 0$, then the theorem is obvious for the function $\psi_+(M)$. We therefore assume that $\|\psi_+\| > 0$. Fix a number ε_0, $0 < \varepsilon_0 < \|\psi_+\|$. By the construction of $\psi_+(M)$, there exists a strip γ_1,

bounded by the curve Γ and by a simple closed curve $\Gamma_1 \subset D$, such that $\psi_+(M) \leqslant \varepsilon_0,\ M \in \Gamma + \gamma_1 + \Gamma_1$.

Define on $D + \Gamma$ the functions $\psi_1(M)$ and $\psi_2(M)$, putting

$$\psi_1(M) = \begin{cases} \psi_+(M), & M \in \Gamma + \gamma_1 + \Gamma_1, \\ 0, & M \in D - \gamma_1 - \Gamma_1; \end{cases} \qquad \psi_2(M) = \begin{cases} 0, & M \in \Gamma + \gamma_1 + \Gamma_1, \\ \psi_+(M), & M \in D - \gamma_1 - \Gamma_1. \end{cases}$$

We have

$$\psi_+(M) = \psi_1(M) + \psi_2(M), \qquad \psi_1(M) \leqslant \varepsilon_0, \qquad M \in D + \Gamma,$$

whence

$$\|A^n\psi_+\| \leqslant \|A^n\psi_1\| + \|A^n\psi_2\|, \qquad \|A^n\psi_1\| \leqslant \varepsilon_0. \tag{4}$$

If it is proved that $\lim \|A^n\psi_2\| \leqslant \varepsilon_0$, then, taking (4) into account, we obtain

$$0 \leqslant \lim \|A^n\psi_+\| \leqslant 2\varepsilon_0,$$

and since ε_0 is any number satisfying the condition $0 < \varepsilon_0 < \|\psi_+\|$, the relation (2) will thereby be proved. Here the convergence, for example, of the sequence $\{\|A^n\psi_2\|\}$ follows from the fact that it is nonincreasing and bounded below.

Thus, it remains to prove the validity of the inequality

$$\lim \|A^n\psi_2\| \leqslant \varepsilon_0.$$

For this purpose introduce a function $\lambda(M)$, continuous on $\Gamma + \gamma_1 + \Gamma_1$ and harmonic in γ_1, satisfying the conditions

$$\lambda(M) = 0, \quad M \in \Gamma, \qquad \lambda(M) = \|\psi_+\|, \quad M \in \Gamma_1,$$

and define on $D + \Gamma$ the function $\tilde{\lambda}(M)$, putting

$$\tilde{\lambda}(M) = \begin{cases} \lambda(M), & M \in \Gamma + \gamma_1 + \Gamma_1, \\ \|\psi_+\|, & M \in D - \gamma_1 - \Gamma_1. \end{cases}$$

Next find a constant $K_0 > 1$ such that the level line $\|\psi_+\|/K_0$ of the function $\tilde{\lambda}(M)$ (denote it by Γ_2) bounds, together with the curve Γ, a domain (denote this domain by γ_2) possessing, along with Γ_2, the following property. If $M \in \gamma_2 + \Gamma_2$, then $C_{h(M)} \subset \Gamma + \gamma_1 + \Gamma_1$. Let us note that Γ_2 is a simple closed curve, as is easily seen from the properties of the function $\tilde{\lambda}(M)$ harmonic in γ_1.

Finally define on $D + \Gamma$ the function $\Phi(M)$, setting

$$\Phi(M) = \begin{cases} K_0\tilde{\lambda}(M), & M \in \Gamma + \gamma_2 + \Gamma_2, \\ \|\psi_+\|, & M \in D - \gamma_2 - \Gamma_2. \end{cases}$$

The function $\Phi(M)$ is continuous on $D + \Gamma$ and harmonic in γ_2. It is not difficult to see that

$$A\Phi(M) \leqslant \Phi(M), \qquad M \in D + \Gamma,$$

whence, by virtue of property 3 of the averaging operator,

$$A^n\Phi(M) \leqslant A^{n-1}\Phi(M) \leqslant \cdots \leqslant A\Phi(M) \leqslant \Phi(M), \qquad M \in D + \Gamma.$$

Recalling the definition of the function $\psi_2(M)$, we find

$$\psi_2(M) \leqslant \Phi(M), \qquad M \in D + \Gamma.$$

Therefore

$$A^n\psi_2(M) \leqslant A^n\Phi(M) \leqslant \Phi(M), \qquad M \in D + \Gamma,$$

i.e., $\Phi(M)$ is a majorant of the functions of the sequence $\{A^n\psi_2(M)\}$.

Denoting by Γ_{ε_0} the level line ε_0 of the function $\Phi(M)$, and by γ_{ε_0} the domain bounded by the curves Γ and Γ_{ε_0}, we find

$$A^n\psi_2(M) \leq \Phi(M) \leq \varepsilon_0, \qquad M \in \Gamma + \gamma_{\varepsilon_0} + \Gamma_{\varepsilon_0}. \tag{5}$$

Let us estimate $\|A^n\psi_2\|$. By virtue of (3), the circle $C_{h(M)}$, $M \in D - \gamma_{\varepsilon_0} - \Gamma_{\varepsilon_0}$, has a nonempty intersection with the set $\Gamma + \gamma_{\varepsilon_0} + \Gamma_{\varepsilon_0}$, on which, according to (5), the functions of the sequence $\{A^n\psi_2(M)\}$ are bounded by the maximum number ε_0. Making simple transformations, we therefore find

$$\|A^n\psi_2\| \leq \|\psi_+\| - \alpha_{n-1}\|\psi_+\| - \varepsilon_0|, \tag{6}$$

where

$$\alpha_n = \alpha_0 \left[1 + \sum_{i=1}^{n} (1-\alpha_0)^i \right], \qquad \alpha_0 = \frac{\min\limits_{M \in D - \gamma_{\varepsilon_0}} \operatorname{mes} C_{h(M)} \cap \gamma_{\varepsilon_0}}{\pi (d(D)/2)^2}, \qquad 0 < \alpha_0 < 1,$$

and by $d(D)$ is denoted the diameter of the domain D. In this case the sequence $\{\alpha_n\}$ obviously increases, and $\lim \alpha_n = 1$, which together with (6) leads to the inequality $\lim \|A^n \psi_2\| \leq \varepsilon_0$, which was to be obtained.

We note that Theorem 1 is also valid for a multiply connected domain bounded by simple closed curves.

Let us dwell further on results that hold in the case of one variable. Let $[a,b]$ be a certain interval on the line, and let the function $f(x)$ be continuous on $[a,b]$. The averaging operator by means of which $f(x)$ is associated with the function $Af(x)$ is introduced as follows:

$$Af(a) = f(a), \qquad Af(b) = f(b),$$

$$Af(x) = \frac{1}{\operatorname{mes} C_{h(x)}} \int_{C_{h(x)}} f(t)\, dt, \qquad x \in (a,b),$$

where $C_{h(x)}$ denotes the interval $[x-h(x),\ x+h(x)]$, and $h(x)$ denotes a function continuous on $[a,b]$ such that: a) $h(x) > 0$, $x \in (a,b)$; b) $C_{h(x)} \subset [a,b]$, $x \in [a,b]$.

Let the function $\chi(x)$ be linear on $[a,b]$, and let $\chi(a) = f(a)$, $\chi(b) = f(a)$. As in the case of two variables, the theorem on the uniform convergence on $[a,b]$ of the sequence of functions $A^n f(x)$ to $\chi(x)$ is valid.

Theorem 2. Let the function $f(x)$ be twice continuously differentiable and convex (concave) on $[a,b]$; then the function $Af(x)$ is also twice continuously differentiable and convex (concave) on $[a,b]$, if

$$h(x) = \frac{-x^2 + x(a+b) - ab}{b-a}, \qquad x \in [a,b]. \tag{7}$$

Let the symbol $D[f(x)]$ denote the Dirichlet integral of the function $f(x)$, i.e.

$$D[f(x)] = \int_a^b |f'(x)|^2\, dx.$$

Theorem 3. If the function $f(x)$ is twice continuously differentiable on $[a,b]$ and the averaging radius $h(x)$ is given by formula (7), then

$$D[A^n f(x)] \to D[\chi(x)].$$

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Note: Figure translations are in progress. See original paper for figures.