MATHEMATICS

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Abstract

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MATHEMATICS

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ON THE FIRST BOUNDARY VALUE PROBLEM FOR SOME SEMIBOUNDED OPERATORS

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I. Let

$$A(x,\xi)=\sum_{\alpha} a_{\alpha}(x)\xi^{\alpha} \tag{1}$$

be a polynomial in $\xi=(\xi_1,\dots,\xi_n)$ with real coefficients $a_\alpha(x)$, depending on the parameter $x=(x_1,\dots,x_n)\in\overline{Q}$, where Q is some domain of n-dimensional x-space, $\alpha=(\alpha_1,\dots,\alpha_n)$, $\xi^\alpha=\xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}$.

Consider the set $E(x)$ of integer points α lying in the region $\alpha_1\geqslant 0,\dots,\alpha_n\geqslant 0$ and such that $a_\alpha(x)\neq 0$. Consider also the minimal convex polyhedron $G(x)$ containing $E(x)$. Denote the union of $E(x)$ and $G(x)$ over all $x\in\overline{Q}$ by E and G, respectively. Let $e_1,\dots,e_N$, $e_i=(e_i^1,\dots,e_i^n)$, $i=1,\dots,N$, be the set of vertices of G, and let $\Gamma_1,\dots,\Gamma_M$ be the set of its $(n-1)$-dimensional faces.

Consider the face Γ_k with exterior normal $m_k=(m_k^1,\dots,m_k^n)$, $k=1,\dots,M$. Let

$$A_{\Gamma_k}(\xi)=\sum_{\alpha\in\Gamma_k} a_\alpha(x)(i\xi)^\alpha.$$

Since $\alpha\in\Gamma_k$, $A_{\Gamma_k}(\xi)$ is a generalized homogeneous polynomial in ξ.* Take a vector $c=(c_1,\dots,c_n)$, whose components c_r, $r=1,\dots,n$, are equal either to $+1$ or to -1 (there are 2^n such vectors in all), and make the change of variables

$$\xi_r=c_r\zeta_r^{m_k^r},\quad r=1,\dots,n,$$

where $-\infty<\zeta_r<\infty$, $r=1,\dots,n$ (we shall assume, without loss of generality, that all coordinates of the normal m_k are integers).

Under such a change of variables, $A_{\Gamma_k}(\xi)$ becomes a homogeneous polynomial $\widetilde{A}^{(c)}_{\Gamma_k}(\zeta)$ (the superscript c indicates that a change of variables has been made with some vector c). Represent $\widetilde{A}^{(c)}_{\Gamma_k}(\zeta)$ in the form

$$\widetilde{A}^{(c)}_{\Gamma_k}(\zeta) = \zeta_1^{\rho_1} \cdots \zeta_n^{\rho_n} \,\widetilde{\widetilde{A}}^{(c)}_{\Gamma_k}(\zeta),$$

where $\rho_1, \dots, \rho_n$ are certain integers $\rho_i \geqslant 0$, $i = 1, \dots, n$, and $\widetilde{\widetilde{A}}^{(c)}_{\Gamma_k}(\zeta)$ is a polynomial homogeneous in ζ, whose terms have no common factors of the form $\zeta_r^{\theta_r}$ with $\theta_r > 0$, $r = 1, \dots, n$.

Definition 1. A face Γ_k, $k = 1, \dots, M$, will be called **principal** if its (exterior) normal m_k makes an acute angle with at least one of the coordinate axes Oa_s, $s = 1, \dots, n$.

Definition 2. A face Γ_k, $k = 1, \dots, M$, will be called **leading** if the homogeneous polynomials $\widetilde{\widetilde{A}}^{(c)}_{\Gamma_k}(\zeta)$, for all choices of c, do not vanish on the unit sphere $|\zeta| = 1$.

Definition 3. A vertex e_p, $p = 1, \dots, N$, will be called **principal** if it belongs to at least one principal face.

Let the principal faces of G be the faces $\Gamma_1, \dots, \Gamma_{M_1}$, $M_1 \leqslant M$, and the principal vertices be the vertices $e_1, \dots, e_{N_1}$, $N_1 \leqslant N$.

* For brevity of exposition, it is everywhere assumed in the note that $(m_k, i_s) \neq 0$, $k = 1, \dots, M$, $s = 1, \dots, n$, where i_s is the direction vector of the axis Oa_s.

Assumptions.

1. We shall assume that the intersection E_0 of all $E(x)$, $x \in \overline{Q}$, is nonempty and that all principal vertices $e_1, \dots, e_{N_1}$ of G lie in E_0.

2. For all $x \in \overline{Q}$,

$$\lim_{\xi \to \infty} A(x, \xi) = \infty \tag{2}$$

for real ξ.

3. All principal faces are leading.

Theorem 1. There exists a constant $\gamma_0 > 0$ such that for all $x \in \overline{Q}$ and $\xi \to \infty$,

$$\gamma_0^{-1} \leq \frac{A(x, \xi)}{\xi^{e_1} + \cdots + \xi^{e_N}} \leq \gamma_0.$$

II. Consider the differential operator

$$H(x, D) = \sum_{\alpha} h_\alpha(x) D^\alpha \tag{3}$$

$x = (x_1, \dots, x_n) \in \overline{Q}$, where Q is some bounded domain of the n-dimensional space x, $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $D_i = \partial/\partial x_i$, $i = 1, \dots, n$; the coefficients $h_\alpha(x)$ will be assumed to be complex-valued functions from $C^\infty(\overline{Q})$. The sets $E(x)$, $G(x)$, $E, \dots$, etc., for the operator (3) are introduced in exactly the same way as the corresponding sets in the preceding section (in the real case). We shall assume that, for $H(x, D)$, assumptions 1 and 3 are satisfied, made with respect to the operator $A(x, -iD)$. Then the following holds.

Theorem 2. In order that the operator (3) be hypoelliptic, it is necessary and sufficient that:

1) for real $\xi \to \infty$,
$$H(x, i\xi) \to \infty \quad \text{for } x \in \overline{Q};$$
2) on each coordinate axis $O\alpha_s$, $s = 1, \dots, n$, of the space $\{\alpha\}$ there must lie a principal vertex of the polyhedron G;
3) the (outer) normals of the principal faces of G must have only positive coordinates.

This theorem is proved with the aid of Hörmander' s results ([1]) for hypoelliptic equations with variable coefficients and Theorem 1. Let us note that the necessity of the second condition of Theorem 2 was established in the paper ([2]).

III. Consider, in a domain Q of the n-dimensional space $x = (x_1, \dots, x_n)$, the linear differential equation

$$P_1(x, D)u \equiv \sum_\alpha c_\alpha(x) D^\alpha u = f(x) \tag{4}$$

with sufficiently smooth complex coefficients $c_\alpha(x)$. Suppose that the characteristic polynomial for the operator $P_1(x, D)$ has the form

$$P_1(x, i\xi) = \operatorname{Re} P_1(x, i\xi) + i \operatorname{Im} P_1(x, i\xi) \equiv A(x, \xi) + iB(x, \xi),$$

$$A(x, \xi) = \sum_\alpha a_\alpha(x)\xi^\alpha, \qquad B(x, \xi) = \sum_\alpha b_\alpha(x)\xi^\alpha, \tag{5}$$

where $\xi = (\xi_1, \dots, \xi_n)$ is a real vector, and the polynomial $A(x, \xi)$ is the polynomial from Section I.

In addition to the polyhedron G constructed for the polynomial $A(x, \xi)$, consider also the polyhedron

$$G_1 = \left(\bigcup_{\alpha \in G} R_\alpha \right) \cup G,$$

where R_α is the set of points $\beta = (\beta_1, \dots, \beta_n)$ for which $0 \le \beta_i \le \alpha_i$, $i = 1, \dots, n$. Obviously, G_1 is a convex polyhedron, with n of its faces (not principal ones) lying on the coordinate planes. Denote by $f_1, \dots, f_{N_2}$ the principal vertices of the polyhedron G_1; it turns out that all numbers $f_k^1, \dots, f_k^n$, $k = 1, \dots, N_2$ ($f_k = (f_k^1, \dots, f_k^n)$), are even.

Lemma 1. There exists $\lambda_0 \geq 0$ such that the quadratic norm

$$[u, u] = ((A(x, D) + \lambda_0)u, u) \tag{6}$$

one may take for the square of the norm in $C_0^\infty(Q)$ (the parentheses (,) denote the scalar product in $\mathcal{L}_2(Q)$).

In the Hilbert space $\mathcal{H}_A(Q)$, obtained by completing $C_0^\infty(Q)$ in the metric (6), the scalar product may also be introduced in the following way:

$$[u, v]_1 = \int_Q \sum_{i=1}^{N_2} D^{f_i/2} u \, D^{f_i/2} v \, dx.$$

Functions from $\mathcal{H}_A(Q)$ satisfy on the boundary S of the domain Q certain homogeneous boundary conditions. For brevity, we describe these conditions for the case when S contains no planar pieces parallel to the coordinate planes.

Lemma 2. If $u \in \mathcal{H}_A(Q)$, then

$$D^\beta u\big|_S = 0 \tag{7}$$

for any integer vector β which satisfies at least one of the inequalities $f_r - \beta > 0$, $r = 1, \dots, N_2$ (the vector $\beta \geq 0$ if all its coordinates are nonnegative; the vector $\beta > 0$ if all its coordinates are nonnegative and at least one coordinate is strictly positive). Equality (7) is understood in the mean.

Lemma 3. Let every point α corresponding to $b_\alpha(x) \neq 0$ in Q (see (5)) either be an interior point of G_1 or, being a boundary point, not lie on any of the principal faces of G_1. Then there exist constants $\lambda_1 \geq 0$ and $\lambda_2 > 0$ such that

$$\lambda_2^{-1}[u, u] \leq |(P_1(x, D)u + \lambda_1 u, u)| \leq \lambda_2 [u, u] \tag{8}$$

for all $u \in C_0^\infty(Q)$.

In the case when the coefficients $c_\alpha(x)$ of the operator $P(x, D)$ are real, a stronger statement holds.

Lemma 3′. There exist constants $\lambda_1 \geq 0$ and $\lambda_2 > 0$ such that inequalities (8) hold for any $u \in C_0^\infty(Q)$, if the $c_\alpha(x)$ in (4) are real and every α for which $c_\alpha(x) \neq 0$ in Q satisfies one of the following conditions:

a) α lies inside G_1;

b) α, being on the boundary of G_1, does not lie on any of its principal faces;

c) α lies on the boundary of G_1 or even outside G_1, but $\alpha - i_k$ (if $\alpha - i_k \geq 0$) for all $k = 1, \dots, n$ lie either inside G_1 or on the boundary of G_1, but not on a principal face (here $i_k = (0, \dots, 0, 1, 0, \dots, 0)$).

These lemmas are analogous to the corresponding lemma in ([3]) and are an extension to the operators considered here of Gårding' s well-known lemma ([4]) for elliptic operators.

Definition. A function $u(x) \in \mathcal{H}_A(Q)$ will be called a **generalized solution of the first boundary-value problem for equation** (4), if the integral identity

$$(u, P^*(x, D)v) = (f, v) \tag{9}$$

($f \in \mathcal{L}_2(Q)$, $P^*(x, D)$ is the operator formally adjoint to the operator $P(x, D)$) holds for every function $v \in C_0^\infty(Q)$.

Theorem 3. The problem of finding a generalized solution of the first boundary-value problem for equation (4) is Fredholm if the conditions of Lemma 3 (or 3′) are fulfilled. For the operator $P(x, D) + \lambda$, with λ sufficiently large, the first boundary-value problem is uniquely solvable.

IV. We carry out in the $(n+1)$-dimensional space $\{a\} = \{a_0, a_1, \dots, a_n\}$ the following construction: each point of the boundary Δ_1 of the convex polyhedron G_1, lying in the plane $a_0 = 0$ (G_1 was constructed in Sec. III), is joined to the point $(2\nu_0 + 1, 0, \dots, 0)$, where ν_0 is an integer $\nu_0 \geq 0$ such that

$$2\nu_0 + 1 < \sigma = \max_{1 \leq i \leq n} \sup_{(0, a_1, \dots, a_n) \in G_1} a_i$$

(we assume, for definiteness–

that $\sup \alpha_i = \sigma$ for $i = 1, \dots, i_0$ and $\sup \alpha_i < \sigma$ for $i > i_0$. The conical surface Δ_2 constructed in this way (taken together with Δ_1) will be taken as the lateral surface of the pyramid G_2, whose lower base will be taken to be G_1. The pyramid G_2 is a convex polyhedron in the space $\{\alpha\}$. We shall denote its principal faces by Δ_2^i, $i = 1, \dots, M_1$. Analogously one constructs the polyhedron G_3 with the same base as G_2, but with vertex $(2\nu_0, 0, \dots, 0)$. Suppose that in the domain Q_2, situated in the $(n+1)$-dimensional space $\{x\} = \{x_0, x_1, \dots, x_n\}$ and bounded by the closed surface S_2, the differential equation

$$P_2(x, D)u \equiv \sum_{\alpha \in \Delta_2} r_\alpha(x) D^\alpha u + \sum_{\alpha \in G_3} r_\alpha(x) D^\alpha u = f(x) \tag{10}$$

is given, with real (for simplicity) sufficiently smooth coefficients. Suppose that

$$r_{2\nu_0+1,0,\dots,0}(x) \neq 0 \quad \text{for } x \in \overline{Q}_2; \tag{11}$$

$$\operatorname{Re} \sum_{\alpha \in \Delta_2} r_\alpha(x)(i\xi)^\alpha \geq \gamma^2 \sum_{k=1}^{N_1} |\xi|^{\tilde{e}_k} \quad \text{for } x \in \overline{Q}_2,\ \gamma > 0, \tag{12}$$

where $\tilde{e}_k$ are the vertices of the minimal convex polyhedra lying on Δ_2^i, $i = 1, \dots, M_1$, and containing all points $\alpha \in \Delta_2$ such that $r_\alpha(x) \neq 0$ for $x \in \overline{Q}$.

Suppose also that S_2 has only two points $B = (x_0^b, x_1^b, \dots, x_n^b)$ and $H = (x_0^H, \dots, x_n^H)$ at which the tangent plane to S_2 is perpendicular to the axis Ox_0. Denote by $\mathcal{H}_{P_2}(Q_2)$ the space obtained by completing $C_0^\infty(Q_2)$ in the metric

$$\{u, u\} = \int_{Q_2} \left[(D_0^{\nu_0} u)^2 + \sum (D^{\tilde{e}_i/2} u)^2 \right] dx.$$

By a generalized solution in $\mathcal{H}_{P_2}(Q_2)$ of equation (10) we shall mean (for $f \in L_2(Q_2)$) a function $u(x) \in \mathcal{H}_{P_2}(Q_2)$ for which

$$(u, P_2^*(x, D)v) = (f, v)$$

is fulfilled for every $v \in C_0^\infty(Q_2)$ (P_2^* is the operator formally adjoint to P_2).

Theorem 4. *The problem of solving equation (10) in $\mathcal{H}_{P_2}(Q_2)$ is a Fredholm-type problem if the domain Q_2 lies inside one of the cylinders*

$$|x_i - x_i^H| = c\,(x_0 - x_0^H)^{\frac{2\nu_0+1}{\sigma}+\varepsilon}, \quad i = 1, \dots, i_0;\ x_0 \geq x_0^H,$$

and inside one of the cylinders

$$|x_i - x_i^b| = c\,(x_0^b - x_0)^{\frac{2\nu_0+1}{\sigma}+\varepsilon}, \quad i = 1, \dots, i_0;\ x_0 \leq x_0^b,$$

for some $\varepsilon > 0$.

In §§ III and IV problems with homogeneous boundary conditions were considered. The case of nonhomogeneous conditions is considered analogously to the corresponding case in (3) or (5).

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Note: Figure translations are in progress. See original paper for figures.