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D. V. BEKLEMISHEV

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Abstract

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D. V. BEKLEMISHEV

CLASSIFICATION OF PENCILS OF QUADRATIC FORMS IN THREE VARIABLES

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1°. A pencil of quadratic forms is a set of quadratic forms of the form $\rho_\alpha \varphi^\alpha$ ($\alpha = 1, \dots, \sigma$), where φ^α are fixed quadratic forms, called the generators of the pencil. The coefficients of the generators are the components of a tensor Λ^α_{ij}, symmetric in i, j. The lower indices i and j take values from 1 to n, the number of variables on which the forms of the pencil depend. Tensors of this kind occur in a number of problems in geometry, in particular in the theory of surfaces of intermediate number of dimensions; precisely such a tensor is the second fundamental object of a surface (^{1}). It is natural to regard pencils as equivalent if one of them can be transformed into the other by a linear change of variables and a linear transformation of the generators. A classification of pencils based on this notion of equivalence would be of considerable interest for geometry. However, up to now there has been no systematic algebraic investigation of pencils of quadratic forms. Only for pencils with two generators is there a complete classification, based on the theory of elementary divisors (^{2},^{3}).

An arbitrary pencil, generally speaking, is equivalent to a pencil of forms in a smaller number of variables with a smaller number of generators. The number of essential variables and generators coincides with the so-called vertical and horizontal ranks of the pencil. These invariants were introduced (in terms of the theory of surfaces) in a paper by G. F. Laptev (^{4}).

It is natural that the number of essential generators of a pencil cannot exceed $n(n+1)/2$. It turns out that it is enough to construct the classification of pencils with a smaller number of generators, not exceeding $[\frac{1}{2}(n(n+1)/2+1)]$. Namely, together with a pencil Λ, whose coefficients we denote by $\Lambda^\alpha_{ij} \equiv \Lambda^\alpha_P$ (the index P runs through the $n(n+1)/2$ pairs ij), one may consider a pencil of forms in the same number of variables, whose coefficients V^P_μ are obtained as a fundamental system of solutions of the system of linear equations $\Lambda^\alpha_P V^P = 0$. We shall call the pencil V **dual** to the pencil Λ. It is uniquely determined by the pencil Λ and has $\tau = n(n+1)/2 - \sigma$ generators, if σ is the number of essential generators of the pencil Λ. The pencil dual to V is the pencil Λ.

Pencils with a number of generators greater than $[\frac{1}{2}(n(n+1)/2+1)]$ can be classified according to the classes of the corresponding dual pencils. This makes it possible, in the problem of classifying pencils, to assume the number of generators

$$\sigma \leq \left[\tfrac{1}{2}(n(n+1)/2+1)\right].$$

Pencils with the same number of variables and the same number of generators can be distinguished by ranks. The rank of a pencil is the greatest number r for which the tensor

$$\Lambda^{(\alpha_1}_{i_1[j_1}\Lambda^{\alpha_2}_{i_2|j_2}\cdots\Lambda^{\alpha_r)}_{i_r]|j_r},$$

is nonzero. It can be proved that if the rank of a pencil is r, then the pencil contains at least one form of rank r and contains no forms of rank greater than r.

Consider the $n(n+1)/2$-dimensional linear space K of all quadratic forms in n variables. In it a pencil is represented by a linear sub- ield?

equality of δ dimensions. In the space K there acts a representation of the full linear group in n variables. The transformations of this representation may be regarded as those transformations of the full linear group in $n(n+1)/2$ variables which leave invariant all cones formed by forms of rank $\leq r$ $(r = 1, \dots, n-1)$.

Since together with every form φ the pencil also contains all forms of the form $\lambda\varphi$, it is natural to pass from the space K to the projective space $\widetilde{K}$, formed by the one-dimensional subspaces of the space K. In $\widetilde{K}$ the pencil is represented by the plane $P_{\sigma-1}$, and the system of cones of forms of rank $\leq r$ is represented by a system of algebraic surfaces R^r—the "directrices" of these cones. These surfaces are nested one in another:

$$R^{n-1} \supset R^{n-2} \supset \cdots \supset R^1.$$

Each subsequent member of this sequence is the set of singular points of the preceding one (i.e. those points at which all partial derivatives of the left-hand sides of its equations vanish). A pencil of rank r lies in the flat generating surface R^r.

The projective invariants of the intersection $P_{\sigma-1}$ with each of the R^r are invariants of the pencil. In other words, for equivalence of pencils it is necessary that the intersections $P_{\sigma-1}$ with R^r be projectively equivalent. In the general case these invariants cannot be made the basis of a complete classification of pencils. Although for pencils of binary forms and for pencils of ternary forms with two generators such a principle of classification leads to a complete classification, already for pencils of ternary forms with three generators it is necessary to add one more invariant, independent of those under consideration.

2°. Below is given the classification of all pencils of ternary forms. The coefficients are everywhere assumed to be complex. We assume that in each case the number of variables and the number of generators are not reduced.

I. A pencil with 6 generators is the totality of all ternary forms.

II. A pencil with 5 generators has as its dual a pencil with one generator. The only invariant of the latter is the rank. The classification of pencils with one generator, and hence also of pencils with 5 generators, is completely obvious.

III. A pencil with 4 generators has as its dual pencil a pencil with 2 generators. The classification of pencils with 2 generators is known. It is easy to obtain the canonical forms of pencils with 4 generators by finding the duals of the pencils with 2 generators taken in canonical form.

IV. Thus the only case requiring investigation is the case of a pencil with 3 generators. In what follows we speak only of such pencils.

1. Pencils of rank 1 do not exist, since, as can be shown in general, a pencil of rank 1 cannot have more than one generator.
2. Pencils of rank 2 are all equivalent to one another and are reduced to the canonical form with generators $\varphi^1 = x_1^2,\ \varphi^2 = 2x_1x_2,\ \varphi^3 = 2x_1x_3$.
3. A pencil of rank 3 with 3 generators is represented in the space $\widetilde{K}$ by the plane P_2. The intersection of P_2 with R^2 determines in the two-dimensional projective plane P_2 a curve of the 3rd order, which we shall call the D-curve. I use the complex projective classification of plane cubic curves in the form in which it is given in (5).

A. Let us analyze the case when the D-curve has singular points. Subpencils with one generator of rank 1 necessarily correspond to singular points of the D-curve; however, singular points may also correspond to subpencils of rank 2. To each canonical form of the equation of the D-curve there correspond, generally speaking, several canonical forms of the pencil. They differ by the presence and (in one case) by the arrangement of subpencils with one generator of rank 1.

Table 1 gives the canonical forms of pencils whose D-curves have singular points. The canonical form of a pencil is written as a matrix, in whose rows stand the coefficients of the generators in the following order: $\Lambda_{11}, \Lambda_{12}, \Lambda_{13}, \Lambda_{22}, \Lambda_{23}, \Lambda_{33}$.

Table 1

Number	D-curve	Rank-1 subpencils	Canonical form of the pencil
1	$\rho_1^3 = 0$		$\begin{Vmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{Vmatrix}$
2a	$\rho_1^2\rho_2 = 0$	Two, including φ_3—the intersection of the lines $\rho_1 = 0$ and $\rho_2 = 0$	$\begin{Vmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{Vmatrix}$

Number	D-curve	Rank-1 subpencils	Canonical form of the pencil
2	$\rho_1^2\rho_2 = 0$	Two, different from φ_3	$\begin{Vmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{Vmatrix}$
2	$\rho_1^2\rho_2 = 0$	One, different from φ_3	$\begin{Vmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{Vmatrix}$
3	$\rho_1^3 + \rho_2^3 = 0$		The pencil does not exist
4	$\rho_1^2\rho_2 + \rho_2^2\rho_3 = 0$		$\begin{Vmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{Vmatrix}$
5	$\rho_1^3 + 3\rho_1^2\rho_3 = 0$		$\begin{Vmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{Vmatrix}$
6a	$\rho_1\rho_2\rho_3 = 0$	Absent	$\begin{Vmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{Vmatrix}$
6	$\rho_1\rho_2\rho_3 = 0$	Exist	$\begin{Vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{Vmatrix}$
7a	$\rho_1^3 + \rho_1\rho_2\rho_3 = 0$	Absent	$\begin{Vmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{Vmatrix}$
7	$\rho_1^3 + \rho_1\rho_2\rho_3 = 0$	Exist	$\begin{Vmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{Vmatrix}$
8a	$\rho_1^3 + \rho_2^3 + \rho_1\rho_2\rho_3 = 0$	Absent	$\begin{Vmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{Vmatrix}$
8	$\rho_1^3 + \rho_2^3 + \rho_1\rho_2\rho_3 = 0$	Exist	$\begin{Vmatrix} 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{Vmatrix}$

B. Let us now consider the case of a pencil whose D-curve has no singular points. In this case the equation of the D-curve is reduced to the canonical form

$$\rho_1^3 + \rho_2^3 + \rho_3^3 + 6m\rho_1\rho_2\rho_3 = 0$$

where the coefficient m can be found from the invariant of the curve I. As m

one may take any of the roots of the equation:

1. $I(8m^6+20m^3-1)^2+(8m^3+1)^3=0, \quad -1\neq I\neq\infty$; anharmonic lines.
2. $8m^6+20m^3-1=0, \quad I=\infty$; harmonic line.
3. $m^4-m=0, \quad I=-1$; equianharmonic line.

It will be convenient for us to denote $\alpha=m/\sqrt{8m^3+1}$. This is always possible, since $8m^3+1\neq 0$. By a transformation of the variables and of the generators of the pencil

can be brought to one of the following three forms

$$\begin{cases}\varphi_1=x_1^2+2ux_1x_3+x_3^2,\\ \varphi_2=2x_1x_2,\\ \varphi_3=x_2^2+2x_1x_3;\end{cases} \qquad \begin{cases}\psi_1=x_1^2+2\frac{u-3}{u+1}x_1x_3+x_3^2,\\ \psi_2=2x_1x_2,\\ \psi_3=x_2^2+2x_1x_3;\end{cases}$$

$$\begin{cases}\chi_1=x_1^2+2\frac{u+3}{u-1}x_1x_3+x_3^2,\\ \chi_2=2x_1x_2,\\ \chi_3=x_2^2+2x_1x_3;\end{cases} \tag{1}$$

where

$$u=u(p_1)=\frac{\sqrt{-3}\,(p_1+3\alpha^2)}{\sqrt{p_1^2+6\alpha^2p+\alpha}},$$

and p_1 is any root of the equation

$$8p^3+72\alpha^2p^2+6(27\alpha^3+1)\alpha p+27\alpha^3-\frac{1}{4}=0.$$

If p_2 and p_3 are the other two roots of this equation, then

$$u(p_2)=\frac{u(p_1)-3}{u(p_1)+1},$$

$$u(p_3)=\frac{u(p_1)+3}{u(p_1)-1}.$$

It is easy to see that the set of the three pencils (1) does not depend on which root of the equation is chosen.

For anharmonic lines $-1\neq I\neq\infty$, all three forms (1) are inequivalent. To determine to which of these forms the given pencil is reducible, one may compute the Jacobian of the three forms of the pencil with respect to the three variables. This is a cubic form in x_1, x_2, x_3. Its Aronhold invariant must coincide with the same invariant of the corresponding pencil from (1).

For the equianharmonic curve $I = -1$, all three pencils in formula (1) are equivalent. For the harmonic curve $I = \infty$, two of the three pencils are equivalent.

Moscow Institute of Physics and Technology

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Note: Figure translations are in progress. See original paper for figures.