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Abstract

Full Text

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DUALITY OF FUNCTORS

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In the present note a duality of functors will be constructed in a broad class of categories. This class includes, in particular, the categories of abelian groups, topological spaces (without a distinguished point and with a distinguished point), Banach spaces, sets, partially ordered sets, and structures. (As morphisms we take, respectively, homomorphisms, continuous mappings, continuous linear operators, mappings, isotone mappings.) The categories listed, with the exception of the category of Banach spaces, satisfy all the axioms formulated below; for functors in the category of Banach spaces one can define duality, since all necessary requirements are satisfied by the category of unit balls in Banach spaces (by a morphism in it we mean a linear operator of norm $\leqslant 1$). For functors in the category of topological spaces with a distinguished point, the duality defined in the note coincides with the duality constructed by D. B. Fuks ([1]). Some of the assertions formulated below, namely Propositions 3, 4, 5, were proved by D. B. Fuks for this category.

Let $\mathfrak{A}$ be some category. The set of morphisms of an object X into an object Y will be denoted by $\mathrm{Hom}(X, Y)$. The functors under consideration are assumed to be covariant (unless the contrary is stated); the mapping

$$\mathrm{Hom}(X, Y) \to \mathrm{Hom}(F(X), F(Y))$$

generated by a functor F is denoted by $F_{X,Y}$, or simply by F. The totality of functors acting from the category $\mathfrak{A}$ to the category $\mathfrak{B}$ forms a category if a morphism is taken to be a mapping of one functor into another ([2]). A spectrum* in the category $\mathfrak{A}$ is a totality $\{R_\lambda, \Pi_\lambda^{\lambda'}\}$ of objects R_λ of the category $\mathfrak{A}$, supplied with indices from some class Λ, and, possibly empty, sets

$$\Pi_\lambda^{\lambda'} \subset \mathrm{Hom}(R_\lambda, R_{\lambda'})$$

satisfying the condition

$$\Pi_\lambda^{\lambda'} \cdot \Pi_{\lambda'}^{\lambda''} \subset \Pi_\lambda^{\lambda''}.$$

A mapping $\{\pi_\lambda\}$ of a spectrum $\{R_\lambda\}$ into an object T (of an object T into a spectrum $\{R_\lambda\}$) is a totality of mappings

$$\pi_\lambda : R_\lambda \to T \qquad (\pi_\lambda : T \to R_\lambda)$$

such that

$$\pi_{\lambda'} \Pi_\lambda^{\lambda'} = \pi_\lambda \qquad (\pi_{\lambda'} = \Pi_\lambda^{\lambda'} \pi_\lambda).$$

An object R is called the inductive (projective) limit of a spectrum $\{R_\lambda\}$ (we write, respectively,

$$R = \underrightarrow{\lim}\{R_\lambda, \Pi_\lambda^{\lambda'}\}, \qquad R = \underleftarrow{\lim}\{R_\lambda, \Pi_\lambda^{\lambda'}\},$$

or, more simply:

$$R = \underrightarrow{\lim}\, R_\lambda, \qquad R = \underleftarrow{\lim}\, R_\lambda,$$

if a mapping $\{\pi_\lambda\}$ of the spectrum into R (of the object R into the spectrum) is given and for every mapping $\{\tau_\lambda\}$ of the spectrum into any object T (of an object T into the spectrum) one can find a mapping $\varphi : R \to T$ ($\varphi : T \to R$) such that

$$\tau_\lambda = \varphi\pi_\lambda \qquad (\tau_\lambda = \pi_\lambda\varphi).$$

A category $\mathfrak{A}$ is called concrete if to every object A there is assigned a set $\widetilde{A}$, and to every morphism $\varphi : A \to B$ a mapping

$$\widetilde{\varphi} : \widetilde{A} \to \widetilde{B},$$

where the conditions are fulfilled: a) $\widetilde{\varphi\psi} = \widetilde{\varphi} \cdot \widetilde{\psi}$; b) if $\varphi \neq \psi$, then $\widetilde{\varphi} \neq \widetilde{\psi}$ (the correspondence $A \to \widetilde{A}$ is a functor from the category $\mathfrak{A}$ to the category of sets). All the categories listed at the beginning of the note are naturally transformed into concrete ones.

Let $\mathfrak{A}$ be a concrete category. We shall say that $\mathfrak{A}$ is a concrete category with projective limit when the following conditions are satisfied:

A1. If

$$R = \underleftarrow{\lim}\{R_\lambda, \Pi_\lambda^{\lambda'}\},$$

then

$$\widetilde{R} = \underleftarrow{\lim}\{\widetilde{R}_\lambda, \widetilde{\Pi}_\lambda^{\lambda'}\}.$$

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* The concept of a spectrum includes at the same time the concepts of direct and inverse spectra introduced by P. S. Aleksandrov ([4]). On the other hand, it is close to the concept of a subcategory.

A2. If the spectrum of sets $\{\widetilde{R}_\lambda, \widetilde{\Pi}_\lambda^{\lambda'}\}$ has a projective limit, then the spectrum of objects $\{R_\lambda, \Pi_\lambda^{\lambda'}\}$ also has a projective limit.

We proceed to describe the class of categories in which one can define the duality of functors.

Let, in a concrete category with projective limit $\mathfrak{A}$:

D1. A functor $H(X, Y)$ is defined, contravariant in X and satisfying the condition

$$\widetilde{H}(X, Y) = \mathrm{Hom}(X, Y).$$

D2. A symmetric functor $X \otimes Y = Y \otimes X$ is defined, for which*

$$(X \otimes Y, Z) = H(X, H(Y, Z)).$$

The functors $H(A, X)$ and $A \otimes X$ (A a fixed object) will be denoted respectively by $\Omega_A(X)$ and $\Sigma_A(X)$. We shall say that a functor F, acting in the category $\mathfrak{A}$, is admissible if for any objects $X, Y \in \mathfrak{B}$ there is a morphism

$$\mathfrak{F}_{X,Y} : H(X, Y) \to H(F(X), F(Y)),$$

such that $\tilde{\mathfrak{F}}_{X,Y} = \widetilde{F}_{X,Y}$.

D3. The functors Ω_A and Σ_A are admissible.

D4. There exists an object I such that the functor Ω_I is isomorphic to the identity functor ($\Omega_I(X) = X$).

A category in which the conditions listed above are fulfilled will be called a D-category. In what follows in the note only D-categories will be considered. Functors acting in a D-category are always assumed to be admissible. We denote by $F(\mathfrak{A})$ the category of admissible functors in the category $\mathfrak{A}$. Note the following.

Proposition 1. If $R = \lim\limits_{\leftarrow} R_\lambda$, then

$$\lim_{\leftarrow} H(X, R_\lambda) = H(X, R).$$

If $R = \lim\limits_{\rightarrow} R_\lambda$, then

$$\lim_{\leftarrow} H(R_\lambda, Y) = H(R, Y), \qquad \lim_{\rightarrow} R_\lambda \otimes Y = R \otimes Y.$$

If F and G are two functors, then we define the object $H(F, G)$ as the projective limit of the spectrum $\{N_\lambda, \Pi_\lambda^{\lambda'}\}_{\lambda \in \Lambda}$, where Λ is the totality of morphisms of the category $\mathfrak{A}$: if $\lambda \in \mathrm{Hom}(X, Y)$, then

$$N_\lambda = H(F(X), G(Y));$$

$\varphi \in \Pi_\lambda^{\lambda'}$, if there are morphisms

$$\alpha \in \mathrm{Hom}(X', X), \qquad \beta \in \mathrm{Hom}(Y, Y'),$$

such that $\lambda' = \beta\lambda\alpha$,

$$\varphi = H(F(\alpha), F(\beta)) : H(F(X), G(Y)) \to H(F(X'), G(Y')).$$

We note that if the objects $H(F, G)$ and $H(F', G')$ exist and mappings of functors

$$\alpha : F' \to F, \qquad \beta : G \to G'$$

are given, then a morphism is naturally defined:

$$H(\alpha, \beta) : H(F, G) \to H(F', G').$$

Proposition 2. If the class $\mathrm{Hom}(F, G)$ is a set, then the object $H(F, G)$ exists and

$$\widetilde{H}(F, G) = \mathrm{Hom}(F, G).$$

We note the following important assertion.

Proposition 3. For any functor F and any object A there is an isomorphism

$$H(\Omega_A, F) = F(A).$$

Definition. We shall say that a functor G is dual to a functor F, and write $G = DF$, if for any object A we have

$$G(A) = H(F, \Sigma_A),$$

and for any morphism $\varphi \in \mathrm{Hom}(A, B)$ we have

$$G(\varphi) = H(1, \widehat{\varphi}),$$

where

$$\widehat{\varphi} : \Sigma_A \to \Sigma_B$$

is the mapping of the functor Σ_A into Σ_B induced by the morphism

$$\varphi : A \to B,$$

and 1 is the identity mapping.

If the functor G is dual to the functor F, then for any pair of objects A, B a morphism

$$G(A) \to H(F(B), \Sigma_A(B))$$

is defined. (We regard $H(F(B), \Sigma_A(B))$ as an element of the spectrum defining the object $H(F, \Sigma_A)$, corresponding to the identity morphism $B \to B$.) By virtue of the isomorphism

$$H(G(A), H(F(B), \Sigma_A(B))) = H(G(A) \otimes F(B), A \otimes B),$$

thereby a morphism is defined:

$$\chi_{A,B} : G(A) \otimes F(B) \to A \otimes B.$$

Proposition 4. The relation

$$H(R, DF) = H(F, DR)$$

holds.

For each functor F we define the mapping

$$\chi : F \to DDF$$

as the mapping corresponding to the identity mapping

$$DF \to DF.$$

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* The equalities $H(X, Y) = \mathrm{Hom}(X, Y)$ and $H(X \otimes Y, Z) = H(X, H(Y, Z))$ should be understood as isomorphisms of functors.

under the isomorphism $H(F, DDF) = H(DF, DF)$. If the mapping $\chi : F \to DDF$ is an isomorphism of functors, then the functor F is called reflexive. It follows from Proposition 4 that for a reflexive functor F there is an isomorphism $H(R, F) = H(DF, DR)$.

Proposition 5. $D\Omega_A = \Sigma_A, \quad D\Sigma_A = \Omega_A$ and, consequently, the functors Σ_A and Ω_A are reflexive.

Proposition 6. If J is a cointegral object[3], then any object X can be monomorphically mapped into the object J_τ—the direct product of τ copies of the object J (for τ one may take the cardinality of X).

Proposition 7. If the category $\mathfrak{A}$ has a cointegral object, then for every functor in the category $\mathfrak{A}$ there exists a dual functor.

This proposition may be applied, in particular, to all the categories listed at the beginning of the note.

Fix an object J and introduce the notation $H(X, J) = \overline{X}$. For each functor F and each object A, define a morphism $\lambda_A : DF(A) \to \overline{F(\overline{A})}$ as follows. From the relation $\mathrm{Hom}(DF(A), \overline{F(\overline{A})}) = \mathrm{Hom}(DF(A) \otimes F(\overline{A}), J)$ it follows that it is enough to construct a morphism $DF(A) \otimes F(\overline{A}) \to J$; this morphism can be constructed as the composition of the morphism $\varkappa_{A,\overline{A}} : DF(A) \otimes F(\overline{A}) \to A \otimes \overline{A}$ and the morphism $A \otimes \overline{A} \to J$, corresponding to the identity under the isomorphism $H(A \otimes \overline{A}, J) = H(\overline{A}, \overline{A})$ (the morphisms λ_A generate a mapping of the functor DF into the functor $S(A) = \overline{F(\overline{A})}$).

Theorem 1. If J is a cointegral object, then for every object A the morphism

$$\lambda_A : DF(A) \to \overline{F(\overline{A})}$$

is a monomorphism.

An object O will be called a zero object of a category if the set O and the sets $\mathrm{Hom}(O, X)$ and $\mathrm{Hom}(X, O)$ for any object X consist of a single element. If a category has a zero object, then one can define the notion of the kernel of a morphism[3]. An object K is called normally embedded in an object X if it is the kernel of some morphism $X \to Y$.

Proposition 8. In a D-category with a zero object, every morphism has a kernel; if K is the kernel of a morphism $\varphi : X \to Y$, then $H(A, K)$ is the kernel of the morphism

$$H(1, \varphi) : H(A, X) \to H(A, Y).$$

If the object K is normally embedded in the object X, then there exists a factor object X/K such that the object $H(X/K, A)$ is the kernel of the morphism

$$H(X, A) \to H(K, A).$$

An object J of a category having a zero object is called universal if, for every object X, the morphism $\iota_X : X \to \overline{\overline{X}}$ is a normal embedding (by $\overline{X}$ is denoted the object $H(X, J)$, and by ι_X the morphism corresponding to the identity under the isomorphism $H(X, \overline{\overline{X}}) = H(\overline{X}, \overline{X})$). A universal object is always a cointegral object.

Theorem 1′. If J is a universal object, then for every functor F and every object A the morphism

$$\lambda_A : DF(A) \to \overline{F(\overline{A})}$$

is a normal embedding.

In particular, the morphism $\lambda_A : DF(A) \to \overline{F(\overline{A})}$ is an isometry for functors in the category of Banach spaces ($\overline{A}$ is the conjugate space), a monomorphism for functors in the category of abelian groups ($\overline{A}$ is the group dual to the group A in the sense of Pontryagin, considered without topology), and a homeomorphism onto a subset for functors in the category of completely regular topological spaces with a distinguished point ($\overline{A}$ is the space of mappings of the space A into the interval).

By the (inductive or projective) limit of a spectrum of functors $\{F_\lambda\}$ we shall mean the limit in the category $\widehat{F}(\mathfrak{A})$.

Proposition 9. If $E = \varinjlim F_\lambda$ $(F = \varprojlim F_\lambda)$ in the category $F(\mathfrak{A})$, then for every object X we have

$$F(X) = \varinjlim F_\lambda(X) \quad (F(X) =$$

$$= \varprojlim F_\lambda(X)).$$

Conversely, if for every object X the spectrum $\{F_\lambda(X)\}$ has an inductive (projective) limit, then the spectrum of functors $\{F_\lambda\}$ also has an inductive (projective) limit.

Proposition 10. If

$$F = \varinjlim F_\lambda,$$

then

$$DF = \varprojlim DF_\lambda.$$

Theorem 2. Let the category $\mathfrak{A}$ satisfy the following condition:

(A) If $\{\pi_\lambda\}$ is a mapping of the spectrum $\{R_\lambda, \Pi_\lambda^{\lambda'}\}$ into an object R, and

$$\widetilde{R} = \varinjlim \{\widetilde{R}_\lambda, \widetilde{\Pi}_\lambda^{\lambda'}\},$$

then

$$R = \varprojlim \{R_\lambda, \Pi_\lambda^{\lambda'}\}.$$

Then for every functor F (reflexive functor F) there exists a spectrum $\{A_\lambda\}_{\lambda\in\Lambda}$ of objects of the category $\mathfrak{A}$ (Λ is some set) such that

$$F = \varprojlim \Omega_{A_\lambda} \qquad (F = \varprojlim \Sigma_{A_\lambda}).$$

The spectrum $\{A_\lambda\}$ generates spectra of functors Ω_{A_λ} and Σ_{A_λ} by virtue of the relations

$$\operatorname{Hom}(\Omega_{A_\lambda}, \Omega_{A_\mu}) = \operatorname{Hom}(A_\mu, A_\lambda), \qquad \operatorname{Hom}(\Sigma_{A_\lambda}, \Sigma_{A_\mu}) = \operatorname{Hom}(A_\lambda, A_\mu).$$

Condition (A) is satisfied, in particular, for the category of abelian groups and for the category of Banach spaces (more precisely, for the category of unit balls in Banach spaces).

Theorem 2′. Let the category $\mathfrak{A}$ satisfy the following condition:

(A') If $\{\pi_\lambda\}$ is a mapping of the spectrum $\{R_\lambda, \Pi_\lambda^{\lambda'}\}$ into an object R, and

$$\widetilde{R} = \varinjlim \{\widetilde{R}_\lambda, \widetilde{\Pi}_\lambda^{\lambda'}\},$$

then there exists

$$\varprojlim \{R_\lambda, \Pi_\lambda^{\lambda'}\} = S$$

and

$$\widetilde{S} = \widetilde{R}.$$

Then every functor F is an inductive limit of a spectrum of functors of the form

$$\Sigma_B \Omega_A.$$

Condition (A') is satisfied, in particular, for the category of topological spaces (with a distinguished point or without a distinguished point).

Remark 1. Many of the propositions formulated above remain valid if one drops the condition of the existence of the projective limit $A2$.

Remark 2. The definition of duality of functors and most of the assertions listed above can be transferred to functors acting from a subcategory $\mathfrak{B} \subset \mathfrak{A}$ to the category $\mathfrak{A}$.

Remark 3. In constructing duality in the definition of a D-category, instead of the condition of concreteness one may require the fulfillment of the following, weaker, condition: to every object A there is assigned a set $\widetilde{A}$, to every morphism $\varphi : A \to B$ a mapping

$$\widetilde{\varphi} : \widetilde{A} \to \widetilde{B},$$

such that

$$\widetilde{\varphi\psi} = \widetilde{\varphi} \cdot \widetilde{\psi},$$

and, moreover, in the case when φ and ψ are morphisms of an object $H(K, L)$ into an object $H(K', L')$ and the corresponding mappings coincide, then the morphisms φ and ψ also coincide. This generalization makes it possible (under some additional conditions) to construct duality of functors acting in the category of morphisms of a D-category.

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Note: Figure translations are in progress. See original paper for figures.